Math 54 Notes

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Announcements:

Office hours next week in 999 Evans:

- Mon, Fri - 3:00 - 4:30
- Wed. - 10:00 - 11:00

§9.8 Exponential of Matrices

Motivation from:

$$x'(t) = ax(t) \quad x(t) \text{ is a function of t} \quad a \in \mathbb{R}$$

The solution is spanned by e^{at} **Matrix eq:**

$$\vec{x}' = A\vec{x}$$
$$A \; n \times n \text{ matrix}$$
$$\vec{x} = \begin{pmatrix} x_1(t) \\ ... \\ x_n(t) \end{pmatrix}$$

We stil want to use e^{at}

Use power series:

1. Recall, if $a \in \mathbb{R}$, then

$$e^a = 1 + a + \frac{1}{2!}a^2 + \frac{1}{3!}a^3 + ...$$
$$= \sum_{n=0}^{\infty} \frac{1}{n!}a^n$$

2. For$A$ $n \times n$ matrix, define

$$e^A = I + A + \frac{1}{2!}A^2 + \frac{1}{3!}A^3 + ...$$
$$e^{tA} = I + tA + \frac{1}{2!}t^2A^2 + \frac{1}{3!}t^3A^3 + ...$$

This is convergent because of reasons

3. Result: e^A matrix $n \times n$
Remark: In general, entries of e^A are not exponents of entries of A

Example:

$$A = \begin{pmatrix} a & \\ & b \end{pmatrix}$$
$$e^A = I + A + \frac{1}{2!}A^2 + \frac{1}{3!}A^3 + ...$$
$$= \begin{pmatrix} 1 & \\ & 1 \end{pmatrix} + \begin{pmatrix} a & \\ & b \end{pmatrix} + \frac{1}{2!}\begin{pmatrix} a^2 & \\ & b^2 \end{pmatrix} + ...$$
$$= \begin{pmatrix} 1 + a + \frac{1}{2!}a^2 + ... & \\ & 1 + b + \frac{1}{2!}b^2 + ... \end{pmatrix}$$
$$= \begin{pmatrix} e^a & 0 \\ 0 & e^b \end{pmatrix}$$

More generally,

$$A = \begin{pmatrix} a_1 & & \\ & ... & \\ & & a_n \end{pmatrix}$$
$$e^A = \begin{pmatrix} e^{a_1} & & \\ & ... & \\ & & e^{a_n} \end{pmatrix}$$

Example

$$A = \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}$$
$$I = \begin{pmatrix} 1 & \\ & 1 \end{pmatrix}$$
$$e^A = I$$
$$e^I = \begin{pmatrix} e & \\ & e \end{pmatrix} = eI$$

Example:

$$\begin{aligned} A &= \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix} \\ A^2 &= \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix} = 0, \quad A^2 = 0, \quad A^3 = 0 \\ e^A &= I + A = \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix} \end{aligned}$$

Remark:

e^A is computable for any given A, using generalized eigenvalues. However if A is diagonalizable: $A = PDP^{-1}$, $A^k = PD^kP^{-1}$

$$\begin{aligned} e^A &= I + A + \frac{1}{2!}A^2 + \frac{1}{3!}A^3 + ... \\ &= I + PDP^{-1} + \frac{1}{2!}PD^2P^{-1} + \frac{1}{3!}PD^3P^{-1} + ... \\ &= P\left(I + D + \frac{1}{2!}D^2 + \frac{1}{3!}D^3\right)P^{-1} \\ &= Pe^DP^{-1} \end{aligned}$$

Now we go back to

$$\vec{x}' = A\vec{x}$$

$$e^{tA} \text{ is } n \times n \text{ matrix, entries are functions of } t$$

We have

$$\begin{aligned} (e^{tA})' &= \left(I + tA + \frac{1}{2!}t^2A^2 + \frac{1}{3!}t^3A^3\right)' \qquad = 0 + A + tA^2 + \frac{1}{2!}t^2A^3 + ... \\ &= A + \frac{1}{1!}tA^2 + \frac{1}{2!}t^2A^3 + ... \\ &= A\left(I + tA + \frac{1}{2!}t^2A^2 + \frac{1}{3!}t^3A^3 + ...\right) \\ &= Ae^{tA} \end{aligned}$$

Is e^{tA} a solution? No, because solutions are columnns.
The columns of e^{tA} are solutions

$$e^{tA} = \begin{pmatrix} \vec{f}_1(t) & \vec{f}_2(t) & ... & \vec{f}_n(t) \end{pmatrix}$$

$\vec{f}_1(t)$ are columns of e^{tA}, vector functions of t

$$(e^{tA})' = \begin{pmatrix} \vec{f}_1'(t) & \vec{f}_2'(t) & ... & \vec{f}_n'(t) \end{pmatrix}$$
$$Ae^{tA} = \begin{pmatrix} A\vec{f}_1 & A\vec{f}_2 & ... & A\vec{f}_n \end{pmatrix}$$

Then, $(e^{tA})' = Ae^{tA}$ means $\vec{f}_i' = A\vec{f}_i$ for every column.

The columns $\vec{f}_1, ..., \vec{f}_n$ are linearly independent because $e^{tA} \cdot e^{-tA} = I$.

Theorem

The columns of e^{tA} is a basis of the solution set of $\vec{x}' = A\vec{x}$.
A general solution is

$$\vec{c} = \begin{pmatrix} c_1 \\ ... \\ c_n \end{pmatrix}$$
$$\vec{x} = e^{tA}\vec{c}$$
$$= c_1\vec{f}_1 + ... + c_n\vec{f}_n$$

e^{tA} is called the fundamental matrix

Theorem:

If $\vec{x}_1(t), ...\vec{x}_n(t)$ is another basis of the solution set,
Then,

$$e^{tA} = \mathbb{X}(t)\mathbb{X}(0)^{-1}$$

Where $\mathbb{X}(t) = \begin{pmatrix} \vec{x}_1(t) & ... & \vec{x}_n(t) \end{pmatrix}$.
Reason: Both column spaces of $e^{tA}, \mathbb{X}(t)$ are the solution set.

$$\therefore e^{tA} = \mathbb{X}(t)B, B \quad n \times n \text{ invertible matrix}$$

Set $t = 0$

$$e^{0A} = \mathbb{X}(0)B$$
$$B = \mathbb{X}(0)^{-1}$$