Math 54 Notes

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Recall:

For

$$\vec{x}' = A\vec{x}$$
$$\vec{x} = \begin{pmatrix} x_1(t) \\ ... \\ x_n(t) \end{pmatrix}$$
$$A = n \times n \text{ matrix w. constant coefficients}$$

If

$$A\vec{u} = \lambda\vec{u}$$
$$u \neq 0 \text{ vector}$$
$$\lambda \in \mathbb{R}$$

Then

$$e^{\lambda t}\vec{u} \text{ is a solution}$$

Check

$$\begin{aligned}(e^{\lambda t})' &= (e^{\lambda t})'\vec{u} \\ &= \lambda e^{\lambda t}\vec{u} \\ &= A(e^{\lambda t}\vec{u})\end{aligned}$$

Theorem:

If $A\vec{u}_i = \lambda_i\vec{u}_i$, $i = 1, 2, 3...n$, and $\vec{u}_1, ...\vec{u}_n$ are linearly independent, then $e^{\lambda_1 t}\vec{u}_1, ..., e^{\lambda_n t}\vec{u}_n$ is a basis of the solution
Remark: The requirement is equivalent to that A be diagonalizable. This holds immediately if

1. A is symmetric
2. All eigenvalues are real and distinct

Example:

$$A = \begin{pmatrix} 3 & 2 \\ -1 & 0 \end{pmatrix}$$

Solve $\vec{x}' = A\vec{x}$
Solution:

1. Find eigenvalues

$$\begin{aligned} \det(A - \lambda I) &= \det \begin{pmatrix} 3-\lambda & 2 \\ -1 & -\lambda \end{pmatrix} \\ &= (3-\lambda)(-\lambda) + 2 \\ &= \lambda^2 - 3\lambda + 2 = 0 \\ \lambda &= 2, 1 \end{aligned}$$

2. Find eigenvectors

$$\begin{cases} \lambda = 1 \quad A - I = \begin{pmatrix} 2 & 2 \\ -1 & -1 \end{pmatrix} \quad \text{Eigenspace spanned by: } \begin{pmatrix} 1 \\ -1 \end{pmatrix} \\ \lambda = 2 \quad A - 2I = \begin{pmatrix} 1 & 2 \\ -1 & -2 \end{pmatrix} \quad \text{Eigenspace spanned by: } \begin{pmatrix} 2 \\ -1 \end{pmatrix} \end{cases}$$

3. The solution set has basis:

$$e^t \begin{pmatrix} 1 \\ -1 \end{pmatrix}, e^{2t} \begin{pmatrix} 2 \\ -1 \end{pmatrix}$$

4. The general solution is

$$\begin{aligned} \vec{x} &= c_1 e^t \begin{pmatrix} 1 \\ -1 \end{pmatrix} + c_2 e^{2t} \begin{pmatrix} 2 \\ -1 \end{pmatrix} \\ &= c_1 \begin{pmatrix} e^t \\ -e^t \end{pmatrix} + c_2 \begin{pmatrix} 2e^{2t} \\ -e^{2t} \end{pmatrix} \\ c_1, c_2 &\in \mathbb{R} \end{aligned}$$

Remark: Diagonalization of A:

$$A = \begin{pmatrix} 1 & 2 \\ -1 & -1 \end{pmatrix} \begin{pmatrix} 1 & \\ & 2 \end{pmatrix} \begin{pmatrix} 1 & 2 \\ -1 & -1 \end{pmatrix}^{-1}$$

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§9.6 Complex eigenvalues

Consider $\vec{x}' = A\vec{x}$, A is $n \times n$ real coefficient matrix. The eigenvalues are rots of $\det(A - \lambda I) = 0$. These can be imaginary

Say $\lambda = \alpha + i\beta$ is a complex root.
Solving the equation $(A - \lambda I)\vec{u} = \vec{0}$, a linear equation with complex coefficients, yields $\vec{u}$, a complex vector.

Then we still have

$$\begin{aligned} A\vec{u} &= \lambda\vec{u} \\ \lambda &= \alpha + i\beta \quad \alpha, \beta \in \mathbb{R} \\ \vec{u} &= \vec{a} + i\vec{b} \quad \vec{a}, \vec{b} \in \mathbb{R}^n \text{ vectors} \end{aligned}$$

$\vec{a}$ is formed by real part of components of $\vec{u}$
We get a complex solution:

$$\begin{aligned} e^{\lambda t}\vec{u} &= e^{\alpha t + i\beta t}(\vec{a} + i\vec{b}) \\ &= e^{\alpha t}(\cos\beta t + i\sin\beta t)(\vec{a} + i\vec{b}) \\ &= (e^{\alpha t}\cos\beta t)\vec{a} + (e^{\alpha t}\sin\beta t)\vec{b} \quad \text{(real part)} \\ &+ ((e^{\alpha t}\cos\beta t)\vec{a} + (e^{\alpha t}\sin\beta t)\vec{b})i \quad \text{(imaginary part)} \end{aligned}$$

Theorem:

If

$$\begin{gathered} \vec{x}' = A\vec{x} \quad An \times n \text{ real matrix} \\ A\vec{u} = \lambda\vec{u} \quad \lambda = \alpha + i\beta, \vec{u} = \vec{a} + i\vec{b} \end{gathered}$$

Then the following are solutions to the differential equation:

$$\begin{gathered} (e^{\alpha t}\cos\beta t)\vec{a} + (e^{\alpha t}\sin\beta t)\vec{b} \\ +((e^{\alpha t}\cos\beta t)\vec{a} + (e^{\alpha t}\sin\beta t)\vec{b})i \end{gathered}$$

Remark: A is $n \times n$
For each complex root, get above 2 solutions.
For each real root, get 1 solution.
This gives us a basis of the solution set if all roots are simple (multiplicity of each root = 1). For a $n \times n$ matrix, you will have n solutions

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Example:

Solve

$$\vec{x}' = \begin{pmatrix} 1 & 2 \\ -2 & 3 \end{pmatrix} \vec{x}$$

Solution:

1. Compute eigenvalues

$$\begin{aligned} \det(A - \lambda I) &= \det \begin{pmatrix} 1-\lambda & 2 \\ -2 & 1-\lambda \end{pmatrix} \\ 0 &= (\lambda - 1)^2 + 4 \\ \lambda &= 1 \pm 2i \end{aligned}$$

2. We got two roots, but we only need one because they are conjugates. We use $\lambda = 1 + 2i$. Find the eigenvectors:

$$\begin{aligned} A - \lambda I &= \begin{pmatrix} 1 & 2 \\ -2 & 1 \end{pmatrix} - \begin{pmatrix} 1+2i & \\ & 1+2i \end{pmatrix} \\ &= \begin{pmatrix} -2i & 2 \\ -2 & -2i \end{pmatrix} \end{aligned}$$

Use row operations to get:

$$\begin{pmatrix} 1 & i \\ 0 & 0 \end{pmatrix}$$

The eigenvector is $\begin{pmatrix} -i \\ 1 \end{pmatrix} = \begin{pmatrix} 0 \\ 1 \end{pmatrix} + i \begin{pmatrix} -i \\ 0 \end{pmatrix}$ for $\lambda = 1 + 2i$

3. The solution set is spanned by

$$\begin{aligned} e^t \cos 2t \begin{pmatrix} 0 \\ 1 \end{pmatrix} - e^t \sin 2t \begin{pmatrix} -1 \\ 0 \end{pmatrix} &= e^t \begin{pmatrix} \sin 2t \\ cos2t \end{pmatrix} \\ e^t \cos 2t \begin{pmatrix} -1 \\ 0 \end{pmatrix} - e^t \sin 2t \begin{pmatrix} 0 \\ 1 \end{pmatrix} &= e^t \begin{pmatrix} -\cos 2t \\ sin2t \end{pmatrix} \end{aligned}$$

We get the general solution

$$\vec{x} = c_1 e^t \begin{pmatrix} \sin 2t \\ cos2t \end{pmatrix} + c_2 e^t \begin{pmatrix} -\cos 2t \\ sin2t \end{pmatrix}$$