Math 54

Vivian Fang

April 25, 2016

- Final: May 10, 3:10-5:10 p.m.
- Covers all sections, with more weight on differential equations.
- No books, no calculators, can bring a cheat sheet.
- There will be a practice exam.

§9.5 Homogeneous Linear Systems with Constant Coefficients

Goal: To solve the matrix equation $\vec{x}' = A\vec{x}$.

Baby Case:
- $n = 1$
- $x = x(t)$
- $x'(t) = ax(t)$
- $A = [a]$

General solution: $x(t) = ce^{at}$, $c \in \mathbb{R}$

To generalize, there are 2 ways:

1. Try something like e^A, "exp of matrix" (will do on Friday)

2. More elementary way is to try to find a solution of the form:

$$x(t) = e^{\lambda t}\vec{u} \quad \lambda \in \mathbb{R}, \quad \vec{u} = \begin{bmatrix} u_1 \\ u_2 \\ \vdots \\ u_n \end{bmatrix}$$

$$x(t) = \begin{bmatrix} e^{\lambda t}u_1 \\ e^{\lambda t}u_2 \\ \vdots \\ e^{\lambda t}u_n \end{bmatrix}$$

Plug in:

$$\vec{x}'(t) = \begin{bmatrix} \lambda e^{\lambda t}u_1 \\ \lambda e^{\lambda t}u_2 \\ \vdots \\ \lambda e^{\lambda t}u_n \end{bmatrix}$$
$$= \lambda \vec{x}(t)$$

Want:

$$\begin{aligned} \lambda e^{t}\vec{u} &= A(e^{\lambda t}\vec{u}) \\ &= e^{\lambda t}A\vec{u} \end{aligned}$$

Get:

$$A\vec{u} = \lambda \vec{u}$$

Looks like it's related to eigenvalues! This means λ is an eigenvalue, and $\vec{u}$ is an eigenvector associated to λ.

Conclusion:

For $\vec{x}' = A\vec{x}$, if $A\vec{u} = \lambda\vec{u}$, then $e^{\lambda t}\vec{u}$ is a solution of the differential equation.

Theorem: $\vec{x}' = A\vec{x}$. If A has n linearly independent eigenvectors $u_1, u_2, \ldots, u_n$, with $A\vec{u_i} = \lambda \vec{u_i}$, then $e^{\lambda_1 t}\vec{u_1}, \ldots, e^{\lambda_n t}\vec{u_n}$ is a basis of the solution set. The general solution is: $\vec{x} = c_1 e^{\lambda_1 t}\vec{u_1}, \ldots, c_n e^{\lambda_n t}\vec{u_n}$, $c_i \in \mathbb{R}$.
Remark: Condition on A is met $\iff$ A is diagonalizable
e.g. If A is symmetric, then A is always diagonalizable.

Recall: To find eigenvalues, solve: $\det(A - \lambda I) = 0$. To find eigenvectors for eigenvalue λ, solve $(A - \lambda I)\vec{u} = 0$ to get $\vec{u}$.

Remark: Start with $y'' + ay' + by = 0$. There are two ways to approach:

1. Find characteristic polynomial, find roots
2. Convert to matrix equation.