Math 54

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§9.4 Non-homogeneous Linear System

Recall: the equation

$$\vec{x}' = A\vec{x}$$

$$\vec{x} = \begin{bmatrix} x_1(t) \\ x_2(t) \\ \vdots \\ x_3(t) \end{bmatrix} \quad \vec{x}' = \begin{bmatrix} x_1'(t) \\ x_2'(t) \\ \vdots \\ x_3'(t) \end{bmatrix} \quad A = \begin{bmatrix} a_{11}(t) & a_{12}(t) & \dots & a_{1n}(t) \\ \vdots & \vdots & \vdots & \vdots \\ a_{n1}(t) & a_{n2}(t) & \dots & a_{nn}(t) \end{bmatrix}$$

which is in normal form, and is homogeneous.

Non-homogeneous matrix equation:

$$\vec{x}' = A\vec{x} + \vec{f}$$

$$\vec{x} = \begin{bmatrix} x_1(t) \\ x_2(t) \\ \vdots \\ x_3(t) \end{bmatrix} \quad \vec{x}' = \begin{bmatrix} x_1'(t) \\ x_2'(t) \\ \vdots \\ x_3'(t) \end{bmatrix} \quad A = \begin{bmatrix} a_{11}(t) & a_{12}(t) & \dots & a_{1n}(t) \\ \vdots & \vdots & \vdots & \vdots \\ a_{n1}(t) & a_{n2}(t) & \dots & a_{nn}(t) \end{bmatrix} \quad \vec{f} = \begin{bmatrix} f_1(t) \\ f_2(t) \\ \vdots \\ f_3(t) \end{bmatrix}$$

Theorem: The general solution of the non-homogeneous equation is:

particular solution of non-homogeneous + general solution of homogeneous

This is usually in the form:

$$\vec{x} = \vec{x_p} + c_1\vec{x_1} + \ldots + c_n\vec{x_n}$$

where x_p is the particular solution, and $c_1\vec{x_1}+\ldots+c_n\vec{x_n}$ is the general solution.

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e.g. Convert the following equation to a matrix equation:

$$y''' + ty'' - \cos ty = 2e^t$$

Solution: Set $x = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} y \\ y' \\ y'' \end{bmatrix}$

The original equation becomes $x_3' + tx_3 - \cos tx_1 = 2e^t$.
Extra relations:

$$\begin{cases} x_1' = x_2 \\ x_2' = x_3 \end{cases}$$

$$\begin{bmatrix} x_1' \\ x_2' \\ x_3' \end{bmatrix} = \begin{bmatrix} x_2 & +0 \\ x_3 & +0 \\ \cos tx_1 - tx_3 & +2e^t \end{bmatrix} = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ \cos t & 0 & -t \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} + \begin{bmatrix} 0 \\ 0 \\ 2e^t \end{bmatrix}$$

In general,

$$y^{(n)} + a_{n-1}y^{n-1} + \ldots + a_0 y = f$$

is written as

$$\vec{x}' = A\vec{x} + \vec{f}$$

where

$$\vec{x} = \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{bmatrix} = \begin{bmatrix} y \\ y' \\ \vdots \\ y^{(n)} \end{bmatrix} \quad A = \begin{bmatrix} 0 & 1 & 0 & \ldots & 0 \\ 0 & 0 & 1 & \ldots & 0 \\ 0 & 0 & 0 & \ddots & \vdots \\ 0 & 0 & \ddots & 0 & 1 \\ -a_0 & -a_1 & \ldots & -a_{n-2} & -a_{n-1} \end{bmatrix} \quad f = \begin{bmatrix} 0 \\ 0 \\ \vdots \\ 0 \\ f \end{bmatrix}$$

Structure of Solution Sets:
Theorem: Initial Value Problem

$$\begin{cases} \vec{x}' = A\vec{x} + \vec{f} \\ \vec{x}(t_0) = \vec{x_0} \end{cases}$$

where $\begin{bmatrix} x_1(t_0) \\ x_2(t_0) \\ \vdots \\ x_n(t_0) \end{bmatrix}$ is given, A, $\vec{f}$ are continuous at t_0, then there is a unique solution.

<u>Definition</u>: Wronskian For n vector functions

$$\vec{x_1} = \begin{bmatrix} x_{1,1}(t) \\ x_{1,2}(t) \\ \vdots \\ x_{1,n}(t) \end{bmatrix}, \ldots, \vec{x_n} = \begin{bmatrix} x_{n,1}(t) \\ x_{n,2}(t) \\ \vdots \\ x_{n,n}(t) \end{bmatrix}$$

We define the Wronskian as:

$$W(\vec{x_1}, \vec{x_2}, \ldots, \vec{x_n})(t) = \det(\vec{x_1}\ \vec{x_2}\ \ldots\ \vec{x_n})$$

<u>Recall</u>: $\vec{x_1}, \ldots, \vec{x_n}$ are linearly independent if the only constants $c_1, \ldots, c_n$ satisfying $c_1\vec{x_1} + \ldots + c_n\vec{x_n}$ are $c_1 = c_2 = \ldots = c_n = 0$.

<u>Theorem</u>: $\vec{x}' = A\vec{x}$, $\vec{x} = \begin{bmatrix} x_1(t) \\ x_2(t) \\ \vdots \\ x_n(t) \end{bmatrix}$. If $\vec{x_1}, \vec{x_2}, \ldots, \vec{x_n}$ are solutions of the equation, then:

1. $W(\vec{x_1}, \vec{x_2}, \ldots, \vec{x_n})(t) \neq 0$ for some (a single) t_0.
2. $W(\vec{x_1}, \vec{x_2}, \ldots, \vec{x_n})(t) \neq 0$ for every t.
3. $\vec{x_1}, \vec{x_2}, \ldots, \vec{x_n}$ are linearly independent.
4. $\vec{x_1}, \vec{x_2}, \ldots, \vec{x_n}$ form the basis of the solution set.

General Solution:

$$c_1\vec{x_1} + c_2\vec{x_2} + \ldots + c_n\vec{x_n}$$

where $c_1, c_2, \ldots, c_n$ are constants.

Notation book uses:

$$\mathbb{X} = \begin{bmatrix} \vec{x_1} & \vec{x_2} & \ldots & \vec{x_n} \end{bmatrix}$$

The general solution is:

$$\vec{x} = \mathbb{X}\vec{c}$$

And

$$W(\vec{x_1}, \vec{x_2}, \ldots, \vec{x_n})(t) = \det\mathbb{X}$$

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e.g. Verify that the following is a fundamental set of solutions of:

$$\vec{x_1} = \begin{bmatrix} e^{2t} \\ e^{2t} \\ e^{2t} \end{bmatrix}, \vec{x_2} = \begin{bmatrix} -e^{t} \\ 0 \\ e^{-t} \end{bmatrix}, \vec{x_3} = \begin{bmatrix} 0 \\ e^{-t} \\ -e^{t} \end{bmatrix}$$

1. Check that $\vec{x_1}$, $\vec{x_2}$, $\vec{x_3}$ are solutions by plugging in.

2. Compute the Wronskian at $t = 0$.

$$W(\vec{x_1}, \vec{x_2}, \vec{x_3})(0) = \begin{vmatrix} 1 & -1 & 0 \\ 1 & 0 & 1 \\ 1 & 1 & -1 \end{vmatrix} \neq 0.$$