Math 54

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Chapter 9. Jerry Sucks Ass Method

e.g. Find functions $x(t)$, $y(t)$ satisfying

$$\begin{cases} x'(t) = y(t) \\ y'(t) = -x(t) \end{cases}$$

<u>Guess</u>:

$$\begin{cases} x = \sin t \\ y = \cos t \end{cases} \qquad \begin{cases} x = \cos t \\ y = -\sin t \end{cases}$$

or more generally:

$$\begin{cases} x = c_1 \cos t + c_2 \sin t \\ y = -c_1 \sin t + c_2 \cos t \end{cases}$$

This gives all the solutions. It's better to write:

$$\begin{bmatrix} x \\ y \end{bmatrix} = c_1 \begin{bmatrix} \cos t \\ -\sin t \end{bmatrix} + c_2 \begin{bmatrix} \sin t \\ \cos t \end{bmatrix}$$

$$\begin{aligned} &x' = y \\ &x'' = y' = -x \\ &x'' = -x \\ &x = c_1 \cos t + c_2 \sin t \\ &y = -c_1 \sin t + c_2 \cos t \end{aligned}$$

e.g.

$$\begin{cases} x' = x - 2y \\ y' = 3x + 5y \end{cases}$$

t is the variable, $x = x(t)$, $y = y(t)$ are functions of t. Still an ODE.

Rewrite the equation:

$$\begin{bmatrix} x' \\ y' \end{bmatrix} = \begin{bmatrix} x - 2y \\ 3x + 5y \end{bmatrix} = \begin{bmatrix} 1 & -2 \\ 3 & 5 \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix}$$

Notation (equivalent to):

$$\begin{bmatrix} x \\ y \end{bmatrix}' = \begin{bmatrix} 1 & -2 \\ 3 & 5 \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix}$$

If we denote $\vec{u}(t) = \begin{bmatrix} x(t) \\ y(t) \end{bmatrix}$, $A = \begin{bmatrix} 1 & -2 \\ 3 & 5 \end{bmatrix}$, then the equation becomes $\vec{u}' = A\vec{u}$.

Definition: A homogeneous matrix (differential) euqation is of the form:

$$\vec{x}' = A\vec{x}$$

where

$$\vec{x} = \vec{x}(t) = \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{bmatrix} = \begin{bmatrix} x_1(t) \\ x_2(t) \\ \vdots \\ x_n(t) \end{bmatrix}$$

$x_1(t), x_2(t), \ldots, x_n(t)$ are functions of t.

$$\vec{x}' = \vec{x}'(t) = \begin{bmatrix} x_1' \\ x_2' \\ \vdots \\ x_n' \end{bmatrix} = \begin{bmatrix} x_1'(t) \\ x_2'(t) \\ \vdots \\ x_n'(t) \end{bmatrix}$$

$$A = A(t) = \begin{bmatrix} a_{11} & a_{12} & \ldots & a_{1n} \\ \vdots & \vdots & \vdots & \vdots \\ a_{n1} & a_{n2} & \ldots & a_{nn} \end{bmatrix} = \begin{bmatrix} a_{11}(t) & a_{12}(t) & \ldots & a_{1n}(t) \\ \vdots & \vdots & \vdots & \vdots \\ a_{n1}(t) & a_{n2}(t) & \ldots & a_{nn}(t) \end{bmatrix}$$

A is an $n \times n$ matrix, coefficients are functions of t. Called the coefficient matrix. $\vec{x}' = A\vec{x}$ is called the normal form. If the coefficients of A are constants, we say the equation has constant coefficients.

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e.g. Convert the following equation to normal form:

$$\begin{cases} x_1' = x_2 \cos t + 8x_3 \\ x_2' = e^t x_1 - 8t^2 x_2 + x_3 \\ x_3' = x_1 + 5x_2 - 2x_3 \end{cases}$$

Solution:

$$\begin{bmatrix} x_1' \\ x_2' \\ x_3' \end{bmatrix} = \begin{bmatrix} 0 & \cos t & 8 \\ e^t & -8t^2 & 1 \\ 1 & 5 & -2 \end{bmatrix} = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}$$

Set:

$$\vec{x} = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}, A = \begin{bmatrix} 0 & \cos t & 8 \\ e^t & -8t^2 & 1 \\ 1 & 5 & -2 \end{bmatrix}$$

Normal form: $\vec{x}' = A\vec{x}$

Definition: For the matrix equation $\vec{x}' = A\vec{x}$, a solution is a vector function which satisfies the equation for $\vec{x}$. The solution set is the set of all solutions. It is a vector space ("usually" dim $= n =$ number of components of $\vec{x}$).

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e.g. Convert the following equation to a matrix equation:

$$y'' + 2y' + 3y = 0$$

Solution:
Set

$$\begin{cases} x_1 = y \\ x_2 = y' \\ x_2' = y'' \end{cases}$$

The equation becomes:

$$x_2' + 2x_2 + 3x_1 = 0$$
$$x_2' = -2x_2 - 3x_1$$

The relation defining x_1, x_2 give $x_1' = x_2$. Then,

$$\begin{cases} x_1' = x_2 \\ x_2' = -2x_2 - 3x_1 \end{cases}$$

We set:

$$\vec{x} = \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}, A = \begin{bmatrix} 0 & 1 \\ -3 & -2 \end{bmatrix}$$

and get $\vec{x}' = A\vec{x}$.

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e.g.

$$\begin{cases} y''' + y' + z' - 5y + 2z = 0 \\ z'' + y'' - 2z = 0 \end{cases}$$

Solution:
Set $x_1 = y$, $x_2 = y'$, $x_3 = y''$, $x_4 = z$, $x_5 = z'$. Get:

$$\begin{cases} x_3' + x_2 + x_5 - 5x_1 + 2x_4 = 0 \\ x_5' + x_3 - 2x_4 = 0 \end{cases}$$

Together with the givens, we get:

$$\vec{x} = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \\ x_5 \end{bmatrix}, A = \begin{bmatrix} 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & -1 & -1 & 5 & -2 \\ 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & -1 & 2 & 0 \end{bmatrix}$$