Math 54

Vivian Fang

April 18, 2016

§6.2: Homogeneous, Linear, Constant Coefficients Equations

$$a_n y^{(n)} + a_{n-1} y^{(n-1)} + \ldots + a_1 y' + a_0 y = 0 \qquad a_i \in \mathbb{R}, a_n \neq 0$$

<u>We did $n = 2$ before:</u>
<u>Key:</u> Characteristic equation: $a_2 r^2 + a_1 r + a_0 = 0$ (Solution set has dimension 2).
For order n:

$$a_n y^{(n)} + a_{n-1} y^{(n-1)} + \ldots + a_1 y' + a_0 y = 0$$

Try $y = e^{rt}$, where r is a constant to be determined.
Plug in:

$$y^{(k)} = r^k e^{rt}$$

$$\begin{aligned} a_n r^n e^{rt} + a_{n-1} r^{n-1} e^{rt} + \ldots + a_1 r e^{rt} + a_0 e^{rt} &= 0 \\ (a_n r^n + a_{n-1} r^{n-1} + \ldots + a_1 r + a_0) e^{rt} &= 0 \\ a_n r^n + \ldots + a_0 &= 0 \end{aligned}$$

This is the characteristic equation. The roots of the characteristic equation will give a basis $\mathscr{B}$ of the solution set as follows:

1. If r is a real root of multiplicity m, then put:

$$e^{rt}, te^{rt}, t^2 e^{rt}, \ldots, t^{m-1} e^{rt}$$

2. If $r = \alpha + i\beta$ is an imaginary root of multiplicity m, then the conjugate $\bar{r} = \alpha - i\beta$ is also a root of multiplicity m. Put:

$$e^{\alpha t}\cos\beta t, te^{\alpha t}\cos\beta t, \ldots, t^{m-1}e^{\alpha t}\cos\beta t$$

$$e^{\alpha t}\sin\beta t, te^{\alpha t}\sin\beta t, \ldots, t^{m-1}e^{\alpha t}\sin\beta t$$

into the basis $\mathscr{B}$. ($r, \bar{r}$ each with multiplicity $m \to 2m$ functions)

Find these functions for all roots. The set $\mathscr{B}$ is a basis of the solution set.
Remark: Dimension of solution set $= n$.

---

e.g. $y^{(8)} - y^{(7)} - 3y^{(6)} + 5y^{(5)} - 2y^{(4)} = 0$
Solution: Characteristic equation:

$$r^8 - r^7 - 3r^6 + 5r^5 - 2r^4 = 0$$
$$r^4(r^4 - r^3 - 3r^2 + 5r^1 - 2) = 0$$

Guess: $r = 1$ is a root.
Set $(r-1)$ to be a factor of the polynomial:

$$\begin{aligned}
&r^4 - r^3 - 3r^2 + 5r^1 - 2 \\
&= (r^4 - r^3) - 3r^2 + 3r + 2r - 2 \\
&= r^3(r-1) - 3r(r-1) + 2(r-1) \\
&= (r-1)(r^3 - 3r + 2) \\
&= (r-1)(r-1)(r^2 + r - 2) \\
&= (r-1)(r-1)(r-1)(r+2)
\end{aligned}$$

Conclusion for the decomposition:

$$\begin{aligned}
&r^8 - r^7 - 3r^6 + 5r^5 - 2r^4 \\
&= r^4(r-1)^3(r+2)
\end{aligned}$$

$$\text{roots: } r = 0, 1, -2$$
$$\text{mult: } m = 4, 3, 1$$

Basis of solution set:

$$\begin{cases} r = 0 & 1, t, t^2, t^3 \\ r = 1 & e^t, te^t, t^2e^t \\ r = -2 & e^{-2t} \end{cases}$$

General solution:

$$y = c_1 + c_2 t + c_3 t^2 + c_4 t^3 + c_5 e^t + c_6 t e^t + c_7 t^2 e^t + c_8 e^{-2t}$$

Notation:
$D = \frac{d}{dt}$ (taking derivative)
$Dy = y'$, $D^2 y = y''$, ...

$$(3D^3 - 2D + 5)y = 3D^3 y - 2Dy + 5y = 3y''' - 2y'' + 5y$$