Math 54

Vivian Fang

April 15, 2016

§6.1: Higher Order Linear Equations, Basic Theory

e.g.

1. $y'' + 2y' + 3y = e^t$ (second order)
2. $y''''' + 2y' - 5y = e^t \sin t$ (fifth order)
3. $y'''' + e^t \sin t y''' - (t^2 + \ln t)y = e^t$ (fourth order, linear variable coefficients - coefficients of y''' , y are function of t)

Definition: A linear n-th order ODE is of the form:

$$a_n(t)y^{(n)} + a_{n-1}(t)y^{(n-1)} + \ldots + a_1(t)y' + a_0(t)y = b(t)$$
$$(a_n(t) \neq 0)$$

$a_0(t), \ldots, a_n(t)$ are functions of t.
If $a_0(t), \ldots, a_n(t)$ are constant functions, we say the equation has constant coefficients.
If $b(t) = 0$, the equation is homogeneous.
Divide both sides by $a_n(t)$. Get:

$$y^{(n)} + p_{n-1}(t)y^{(n-1)} + \ldots + p_1(t)y' + p_0(t)y = g(t)$$

where

$$\begin{cases} g(t) = \dfrac{b(t)}{a_n(t)} \\ p_i(t) = \dfrac{a_i(t)}{a_n(t)} \end{cases}$$

<u>Theorem</u>: The homogeneous equation:

$$y^{(n)} + p_{n-1}(t)y^{(n-1)} + \ldots + p_1(t)y' + p_0(t)y = 0$$

usually has n linearly independent solutions $y_1, y_2, \ldots, y_n$. The form a basis of the solution set. General solution:

$$y(t) = c_1 y_1(t) + \ldots + c_n y_n(t) \qquad c_1, \ldots, c_n \in \mathbb{R}$$

---

e.g. $y'' + \frac{2}{t}y' + y = 0$
<u>Domain</u>: $t \neq 0$. Second order, variable coefficients.

$$y_1 = \frac{1}{t}\cos t \qquad y_2 = \frac{1}{t}\sin t$$

is the basis of our solution set.

<u>Definition</u>: Wronskian
Given differentiable functions:

$$f_1(t), f_2(t), \ldots, f_n(t)$$

the <u>Wronskian</u> is:

$$W(f_1, \ldots, f_n)(t) = \begin{bmatrix} f_1(t) & f_2(t) & \ldots & f_n(t) \\ f_1'(t) & f_2'(t) & \ldots & f_n'(t) \\ \ldots & \ldots & \ldots & \ldots \\ f_1^{(n-1)}(t) & f_2^{(n-1)}(t) & \ldots & f_n^{(n-1)}(t) \end{bmatrix}$$

If $f_1(t), f_2(t), \ldots, f_n(t)$ are linearly dependent, then $c_1 f_1(t), c_2 f_2(t), \ldots, c_n f_n(t) = 0$. Then,

$$c_1 \begin{bmatrix} f_1 \\ f_1' \\ \ldots \\ f_1^{(n-1)} \end{bmatrix} + \ldots + c_n \begin{bmatrix} f_n \\ f_n' \\ \ldots \\ f_n^{(n-1)} \end{bmatrix}$$

The columns of the matrix are linearly dependent, so.

$$\det = W(f_1, \ldots, f_n)(t) = 0 \quad \forall t$$

Conclusion: Linearly independent equations have

$$\det = W(f_1, \ldots, f_n)(t) \neq 0 \quad \forall t$$

Theorem: $y^{(n)} + p_{n-1}(t)y^{(n-1)} + \ldots + p_1(t)y' + p_0(t)y = 0$
If $y_1, y_2, \ldots, y_n$ are solutions, then the following are equivalent:

1. $y_1, y_2, \ldots, y_n$ is linearly independent
2. $\det = W(f_1, \ldots, f_n)(t) \neq 0 \quad \forall t$
3. $\det = W(f_1, \ldots, f_n)(t) \neq 0$ for some single t_0 in the domain
4. $y_1, y_2, \ldots, y_n$ form a basis of the solution set.

"How can I even make a problem on this?" –Yuan

Theorem: Initial Value Problem

$$y^{(n)} + p_{n-1}(t)y^{(n-1)} + \ldots + p_1(t)y' + p_0(t)y = g(t)$$
$$\left\{ y(t_0) = r_0, y'(t_0) = r_1, \ldots, y^{(n-1)}(t_0) = r_{n-1} \right.$$

has a unique solution if $t_0 \in (a, b)$ open interval (contained in domain of every $p_0(t), \ldots, p_{n-1}(t), g(t)$)

Theorem: Non-homogeneous Equation

$$y = y_p + y_{homog}$$