Math 54

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§4.6: Variation of Parameters

Suggested problems

- Computational problems in the textbook (think before computing)

Goal: Still to find particular solution of $ay'' + by' + cy = f(t)$, $a \neq 0$

Recall: If $f(t)$ is a linear combination of $t^m e^{rt}$ (includes t^m if $r = 0$), $t^m e^{\alpha t} \cos \beta t$ (includes $t^m \cos \beta t$), and $t^m e^{\alpha t} \sin \beta t$, we can use the method of undetermined coefficient (*Guess* a form of y_p).

Variation of parameters: Works for any $f(t)$. The previous methods we have been using are a special case of this!

Setup: $ay'' + by' + cy = f(t)$, $a \neq 0$.

1. Find solution of homogeneous equation, say $y = c_1 y_1(t) + c_2 y_2(t)$ is the general solution of the homogeneous equation.

2. Set $y_p = v_1(t) y_1(t) + v_2(t) y_2(t)$.

   - $y_1(t)$ and $y_2(t)$ are solutions of homogeneous equation.
   - $v_1(t)$ and $v_2(t)$ are functions to be determined.

   Hope y_p is a solution of $ay'' + by' + cy = f(t)$.

3. Plug in:

$$
\begin{aligned}
y_p &= v_1 y_1 + v_2 y_2 \\
y_p' &= (v_1 y_1' + v_2 y_2') + (v_1' y_1 + v_2' y_2)
\end{aligned}
$$

Set $v_1'y_1 + v_2'y_2 = 0$ (to simplify)

Then,

$$\begin{aligned} y_p' &= v_1y_1' + v_2y_2' \\ y_p'' &= (v_1y_1'' + v_2y_2'') + v_1'y_1' + v_2'y_2' \end{aligned}$$

Then,

$$ay'' + by' + cy = v_1(ay_1'' + by_1' + cy_1) + v_2(ay_2'' + by_2' + cy_2) + a(v_1'y_1' + v_2'y_2')$$

But,

$$v_1(ay_1'' + by_1' + cy_1) + v_2(ay_2'' + by_2' + cy_2) = 0$$

So we set

$$a(v_1'y_1' + v_2'y_2') = f(t)$$

Then y_p is a solution if

$$\begin{cases} v_1'y_1 + v_2'y_2 = 0 \\ v_1'y_1' + v_2'y_2' = \frac{1}{a}f(t) \end{cases}$$

This is a system of linear equations of v_1', v_2'.

4. Solve for v_1', v_2' (as functions). Then get v_1, v_2 by integrating.
5. $y_p = v_1y_1 + v_2y_2$ is a solution.

Algorithm (for particular solution of $ay'' + by' + cy = f(t)$):

1. Find the basis y_1, y_2 of solution set of the homogeneous equation.
2. Set $y_p = v_1y_1 + v_2y_2$. Form the equations:

$$\begin{cases} v_1'y_1 + v_2'y_2 = 0 \\ v_1'y_1' + v_2'y_2' = \frac{1}{a}f(t) \end{cases}$$

3. Solve for v_1', v_2'. Get v_1, v_2 by integrating.
4. Plug into $y_p = v_1y_1 + v_2y_2$.

e.g. $y'' + y = \csc t = \frac{1}{\sin t}$
Sol:

1. For $y'' + y = 0$, we have:

$$\begin{aligned} y_1 &= \cos t \\ y_2 &= \sin t \end{aligned}$$

2. Set $y_p = v_1 \cos t + v_2 \sin t$. Get:

$$\begin{cases} v_1' \cos t + v_2' \sin t = 0 & \text{(R1)} \\ -v_1' \sin t + v_2' \cos t = \frac{1}{\sin t} & \text{(R2)} \end{cases}$$

Solve for v_1', v_2':

$$\begin{aligned} \text{(R1)} \cos t - \text{(R2)} \sin t \\ v_1'(\cos^2 t + \sin^2 t) = 1 \\ v_1' = 1 \\ v_2' = \frac{\cos t}{\sin t} = \cot t \end{aligned}$$

By integration, we get

$$\begin{aligned} v_1 &= -t + \text{ constant} \\ v_2 &= \int v_2(t) dt = \int \frac{\cos t}{\sin t} dt \\ &(\text{Set } u = \sin t, du = \cos t dt) \\ &= \ln |u| + \text{constant} \\ &= \ln |\sin t| + \text{constant} \end{aligned}$$

3. Plug in, get:

$$y_p = -t \cos t + \ln |\sin t| \sin t$$