Math 54

Vivian Fang

April 11, 2016

§4.5: Superposition Principle

There is a quiz tomorrow!

<u>Theorem:</u> Let $a \neq 0$. If y_1 is a solution of $ay'' + by' + cy = f(t)$, y_2 is a solution of $ay'' + by' + cy = f_2(t)$, then for $k_1, k_1 \in \mathbb{R}$, $k_1y_1 + k_2y_2$ is a solution of $ay'' + by' + cy' = k_1f(t) + k_2f_2(t)$.

---

e.g. Find a particular solution of $y'' + y = t + 3e^t$.
Sol: For $y'' + y = t$, a particular solution is $y_1 = t$. (General method: try $At + B$).
For $y'' + y = e^t$, a particular solution is $y_2 = \frac{1}{2}e^t$.
Then, a particular solution of $y'' + y = t + 3e^t$ is

$$y_1 + 3y_2$$
$$= t + \frac{3}{2}e^t$$

---

e.g. Find a <u>general solution</u> of $y'' + y = t + 3e^t$.
Sol: General solution is a particular solution + a general solution of the homogeneous equation. We already found y_p.
Homogeneous equation:

$$y'' + y = 0$$

Characteristic equation:

$$\lambda^2 + 1 = 0$$

$$\lambda = \pm i$$

General solution:

$$\begin{aligned} &c_1 e^{0t}\cos t + c_2 e^{0t}\sin t \\ &= c_1\cos t + c_2\sin t \end{aligned}$$

The general solution of the original equation is:

$$\begin{aligned} y &= y_p + y_h \\ &= t + \frac{3}{2}e^t + c_1\cos t + c_2\sin t \end{aligned}$$

---

e.g. Find a solution of $y'' + 2y' + y = e^t + te^t$.
Sol: Original Method:

1. $y'' + 2y' + y = e^t$

   Set $y_1 = Ae^t$, plug in, and find A, which will get you y_1.

2. $y'' + 2y' + y = te^t$

   Set $y_2 = Bte^t + Ce^t$, plug in, find B, C, and y_2.

3. Take $y_p = y_1 + y_2$.

Notice that Ae^t and Ce^t will become the same term.

New method: For $y'' + 2y' + y = e^t + te^t$, try $y_p = Dte^t + Ee^t$, plug in, and find D, E.
Computation goes as:

$$\begin{aligned} y_p'' + 2y_p' + y_p &= (Dte^t + Ee^t)'' + 2(Dte^t + Ee^t)' + (Dte^t + Ee^t) \\ &= (Dte^t + Ee^t) + 2(Dte^t + De^t + Ee^t) \\ &\quad + (Dte^t + De^t + De^t + Ee^t) \\ &= 4Dte^t + (4E + 4D)e^t \end{aligned}$$

Set

$$\begin{cases} 4D = 1 \\ 4D + 4E = 1 \end{cases}$$

Get

$$\begin{cases} D = \frac{1}{4} \\ E = 0 \end{cases}$$

So $y_p = \frac{1}{4}te^t$.

---

e.g. $y'' + 2y' + y = 2e^{2x}\sin x - 3e^{2x}\cos x$
Sol: Take $y_p = Ae^{2x}\sin x + Be^{2x}\cos x$ and plug into the original equation. Find A, B.

---

e.g. $y'' + 2y' + y = 2e^{2t}\sin t + 1 + 2t + 3t^2 - 3te^{2t}\cos t + e^{2t}\cos 2t$
Sol: Divide it into 3 equations:

1. $y'' + 2y' + y = 2e^{2t}\sin t - 3te^{2t}\cos t$
2. $y'' + 2y' + y = 1 + 2t + 3t^2$
3. $y'' + 2y' + y = e^{2t}\cos 2t$

For 1: try $y_1 = (A_1t + A_0)e^{2t}\cos t + (B_1t + B_0)e^{2t}\sin t$
For 2, try $y_2 = c_2t^2 + c_1t + c_0$.
For 3, try $y_3 = De^{2t}\cos 2t + Ee^{2t}\sin 2t$
Final: $y_p = y_1 + y_2 + y_3$.

General Principle: $ay'' + by' + cy = f(t)$
$f(t) =$ a linear combination of $p_1(t)e^{rt}$, $p_2(t)e^{\alpha t}\cos\beta t + p_3(t)e^{\alpha t}\sin\beta t$.
Do $p_1(t)e^{rt}$ and
$p_2(t)e^{\alpha t}\cos\beta t + p_3(t)e^{\alpha t}\sin\beta t$ separately.