Math 54

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§4.4: Non-homogenous Equations (cont.)

Recall: $ay'' + by' + cy = f(t)$.
A general solution is:

$$y = y_p(t) + c_1 z_1(t) + c_2 z_2(t) \text{ where } c_1, c_2 \in \mathbb{R}$$

$$\text{and } y_p \text{ is the particular solution.}$$

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e.g. $y'' + 3y' + 2y = f(t)$, with

$$f(t) = 1, 2t, 3t, e^t, te^{2t}, t^2e^{2t}, e^{3t}\sin t, e^{4y}\cos t$$

Case: $f(t) = e^t$
Try: $y = Ae^t$

$$\begin{aligned} y'' + 3y' + 2y &= (Ae^t)'' + 3(Ae^t)' + 2(Ae^t) \\ &= Ae^t + 3Ae^t + 2Ae^t \\ &= 6Ae^t \end{aligned}$$

We get:

$$\begin{cases} 6A = 1 \end{cases}$$

$$A = \frac{1}{6}$$

$$y_p = \frac{1}{6}e^t$$

Case: $f(t) = te^{2t}$
Problem: $y = Ate^{2t}$ does not work.
Instead, use $y = Ate^{2t} + Be^{2t} \leftarrow$ comes from y'. Then,

$$\begin{aligned} y'' + 3y' + 2y &= (Ate^{2t} + Be^{2t})'' + 3(Ate^{2t} + Be^{2t})' + 2(Ate^{2t} + Be^{2t}) \\ &= 2(Ate^{2t} + Be^{2t}) + 3(2Ate^{2t} + Ae^{2t} + 2Be^{2t}) \\ &\quad + A(4te^{2t} + 2e^{2t}) + 2Ae^{2t} + 4Be^{2t} \\ &= 12Ate^{2t} + (7A + 12B)e^{2t} \end{aligned}$$

We get:

$$\begin{cases} 12A = 1 \\ 7A + 12B = 0 \end{cases}$$

$$\begin{cases} A = \frac{1}{12} \\ B = -\frac{7}{144} \end{cases}$$

$$y_p = \frac{1}{12}te^{2t} - \frac{7}{144}e^{2t}$$

The general solution:

$$y(t) = \frac{1}{12}te^{2t} - \frac{7}{144}e^{2t} + c_1e^{-t} + c_2e^{-2t}$$

Case: $f(t) = t^2e^{2t}$
Try: $y = At^2e^{2t} + Bte^{2t} + ce^{2t}$
Plug in, solve for A, B, and C.

Case: $e^{3t}\sin t$
Try: $y = Ae^{3t}\sin t$
Then,

$$\begin{aligned} y'' + 3y' + 2y &= 2Ae^{3t}\sin t + 3A(e^{3t}\sin t + e^{3t}\cos t) \\ &\quad + 3A(9e^{3t}\sin t + 3e^{3t}\cos t + 3e^{3t}\cos t - e^{3t}\sin t) \\ &= (2A + 9A + 27A - 3A)e^{3t}\sin t + (3A + 9A + 9A)e^{3t}\cos t \end{aligned}$$

We get:

$$\begin{cases} 35A = 1 \\ 21A = 0 \end{cases}$$

which is a contradiction (won't work).

Try: $y = Ae^{3t}\sin t + Be^{3t}\cos t$.
Plug in, find A and B.

Case: $f(t) = e^{4t}\cos t$
Try: $y = Ae^{4t}\cos t + Be^{4t}\sin t$.
e.g. $y'' - y = e^t$
Trying $y = Ae^t$ **doesn't work**. $((e^t)'' - e^t = 0)$
Reason: e^t is the solution of the homogenous equation.
Correct method: We try $y = Ate^t$. Then,

$$\begin{aligned} y'' - y &= A(te^t + 2e^t) - Ate^t \\ &= 2Ae^t \text{ (the } Ate^t \text{ cancel).} \end{aligned}$$

Set $2A = 2$, get $A = \frac{1}{2}$.

Remark: $y = Ate^t + Be^t$ is redundant.

Summary:

$$ay'' + by' + cy = f(t), a \neq 0$$

then

$$a\lambda^2 + b\lambda + c = 0$$

is the characteristic equation.

1. $f(t) = t^m e^{rt}$
   Then, $y_p = t^s(A_m t^m + ... + A_1 t + A_0)e^{rt}$
   where
   $$s = \begin{cases} 0 & \text{if } r \text{ not root of char} \\ 1 & \text{if } r \text{ simple root equation} \\ 2 & \text{if } r \text{ double root} \end{cases}$$
2. $f(t) = t^m e^{\alpha t}\cos\beta t$ or $t^m e^{\alpha t}\sin\beta t$

Then,

$$y_p(t) = t^s(A_m t^m + ... + A_0)e^{\alpha t}\cos\beta t + t^s(B_m t^m + ... + B_0)e^{\alpha t}\sin\beta t$$

$$\begin{cases} 0 & \text{if } \alpha + i\beta \text{ not root of characteristic equation} \\ 1000 & \text{if root.} \end{cases}$$