Math 54

Vivian Fang

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§4.3: Non-homogenous Equations

There is a quiz next Tuesday! Covers §6.5, §7.1, §4.2, §4.3, and §4.4.

e.g.

$$2y'' + 3y' + 4y = 0$$

is homogeneous, but

$$2y'' + 3y' + 4y = \sin t + 1$$

is non-homogeneous

We can compare this with linear equations, where A is an $m \times n$ matrix, $\vec{b} \in \mathbb{R}^m$, and $\vec{x} \in \mathbb{R}^n$.

$$A\vec{x} = \vec{0}$$

is homogeneous, and

$$A\vec{x} = \vec{b}$$

is non-homogeneous.

FACTS:

1. The solution set $A\vec{x} = \vec{0}$ is a subspace of $\mathbb{R}^n$ ($\mathbf{null}(A)$).

2. If $\vec{b} \neq \vec{0}$. the solution set $A\vec{x} = \vec{b}$ is **not** a subspace, because $\vec{0} \notin$ solution set.

However, if $\vec{x_0}$ is a solution of $A\vec{x} = \vec{b}$, then the solution set is $\vec{x_0} + \textbf{null}(A)$. i.e. $\{\vec{x_0} + \vec{u} \mid \vec{u} \in \textbf{null}(A)\}$.
Reason: $A(\vec{x_0} + \vec{u}) = A\vec{x_0} + A\vec{u} = \vec{b} + \vec{0}$. This is a translation of $\textbf{null}(A)$ by $\vec{x_0}$.
Suppose $\{z_1, z_2\}$ is a basis of a solution set of $ay'' + by' + cy = 0$. Then, a general solution of $ay'' + by' + cy = f(t)$ is

$$\begin{aligned} y &= y_p + c_1 z_1 + c_2 z_2 \\ &= y_p(t) + c_1 z_1(t) + c_2 z_2(t) \end{aligned}$$

where $y_p(t)$ is a special particular solution.

Our goal today: find $y_p(t)$.

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e.g. $y'' + y = 2$
Sol: A solution is $y = 2$. Basis of $y'' + y = 0$'s solution set is $\{\cos t, \sin t\}$.
General solution set: $y = 2 + c_1 \cos t + c_2 \sin t$.

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e.g.

$$\begin{cases} y'' + y = 2 \\ y(0) = 1, y'(0) = 5 \end{cases}$$

Sol:
Plugging in y(0):

$$1 = 2 + c_1$$

Plugging in y'(0):

$$\begin{aligned} y' &= -c_1 \sin t + c_2 \cos t \\ 5 &= c_2 \end{aligned}$$

We get:

$$\mathbf{c_1 = -1, c_2 = 5}$$

$$y(t) = 2 - \cos t + 5\sin t$$

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e.g. $y'' + y = e^t$
Sol: $y_p = \frac{1}{2}e^t$. Homogeneous Solution: $\{\sin t, \cos t\}$
$y(t) = \frac{1}{2}e^t + c_1 \cos t + c_2 \sin t$

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e.g. $y'' + y = e^{3t}$
Sol: We try $y = Ae^{3t}$.

$$\begin{aligned} y'' + y &= A \ldots 9e^{3t} + A * e^{3r} \\ &= 10Ae^{3t} \\ A &= \frac{1}{10} \end{aligned}$$

$$y(t) = \frac{1}{10}e^{3t} + c_1 \cos t + c_2 \sin t$$

Method of Undetermined Solutions
e.g. Find particular solution of $y'' + 3y' + 2y = f(t)$ for $f(t) = 1$, $2t$, e^{2t}, te^{2t}, t^2e^{2t}, $e^{3t}\sin t$, and $e^{4t}\cos t$

1. $f(t) = 1$

   Sol: $y_p = \frac{1}{2}$.

2. $f(t) = 2t$

   Try: $y = At$. Get $3A + 2At = 2t$. No solution. So we try $y = At + B$. Get:

$$\begin{aligned} 3A + 2(At + B) \\ 2At + (3A + 2B) \\ 2A = 2 \\ 3A + 2B = 0 \\ A = 1, B = -\frac{3}{2} \end{aligned}$$

3. $f(t) = te^{2t}$

   Try: $y = At^2 + Bt + C$. Get:

   $2A + 3(2At + B) + 2(At^2 + Bt + C)$, will finish on Friday.