ex. $f(t) = e^t \sin t$ function. Is f, f', f'' linearly ind ?

(Do there exst numbers a, b, c that are not all 0 s.t.

$$af(t) + bf'(t) + cf''(t) = 0 ?)$$

sol

$$f(t) = e^t \sin t$$

$$f'(t) = e^t \sin t + e^t \cos t$$

$$f''(t) = e^t \sin t + e^t \cos t + e^t \cos t - e^t \sin t$$

$$= 2e^t \cos t.$$

Not lin. independent.

$$f'(t) = f(t) + \tfrac{1}{2} f''(t)$$

$$\to f'' - 2f' + 2f = 0.$$

Re-interpretation:

$y'' - 2y' + 2y = 0$ has solution $y = e^t \sin t$.

Characteristic eqn: $\lambda^2 - 2\lambda + 2 = 0$.

Roots: $\lambda_1 = 1+i$, $\lambda_2 = 1-i$

Gen sol: $y = c_1 e^t \cos t + c_2 e^t \sin t$

Thm $ay'' + by' + cy = 0$, $a \neq 0$

Characteristic eqn: $a\lambda^2 + b\lambda + c = 0$, 2 roots λ_1, λ_2.

① If $\lambda_1 \neq \lambda_2$ & are real, basis of solution set is $e^{\lambda_1 t}, e^{\lambda_2 t}$.

② If $\lambda_1 = \lambda_2$, basis of solution set is $e^{\lambda_1 t}, te^{\lambda_1 t}$.

③ If λ_1, λ_2 are imaginary, $\left(\alpha = \frac{-b}{2a} \quad \beta = \frac{\sqrt{4ac - b^2}}{a} \right)$

$\lambda_1 = \alpha + i\beta \quad \lambda_2 = \alpha - i\beta$

basis of sol set is:

$e^{\alpha t} \cos(\beta t), e^{\alpha t} \sin(\beta t)$

e.g. $y'' = -y$

sol $\lambda^2 + 1 = 0$, $\lambda = \pm i$ $(\alpha = 0, \beta = 1)$

basis: $e^{0t}\sin(t), e^{0t}\sin(t)$

$y(t) = c_1 \sin t + c_2 \cos t$

Reason: $(\lambda_1, \lambda_2) = (\alpha + i\beta, \alpha - i\beta)$

"Old way": $y = c_1 e^{\lambda_1 t} + c_2 e^{\lambda_2 t}$

Problem: λ_1, not real so what is $e^{\lambda_1 t}$?

Use power series.

$e^z = 1 + z + \frac{1}{2!}z^2 + \frac{1}{3!}z^3 + \dots$

Makes sense if z is complex number.

case $x = 0$:

$e^{iy} = 1 + iy + \frac{1}{2!}(iy)^2 + \dots$

$= 1 + iy + \frac{1}{2!}y^2 - \frac{1}{3!}iy^3 + \dots$

$= \underbrace{1 + \frac{1}{2!}y^2 + \frac{1}{4!}y^4 + \dots}_{\cos y} + i(\underbrace{y - \frac{1}{3!}y^3 + \dots}_{\sin y})$

$e^{iy} = \cos y + i \sin y.$

Back to: $e^{\lambda_1 t} = e^{\alpha t}\cos(\beta t) + ie^{\alpha t}\sin(\beta t)$

$e^{\lambda_2 t} = e^{\alpha t}\cos(\beta t) - ie^{\alpha t}\sin(\beta t)$

Then
$$\left.\begin{array}{l} e^{\alpha t}\cos(\beta t) = \frac{1}{2}(e^{\lambda_1 t} + e^{\lambda_2 t}) \\ e^{\alpha t}\sin(\beta t) = \frac{1}{2i}(e^{\lambda_1 t} + e^{\lambda_2 t}) \end{array}\right\}$$
are real solutions