04/01

§ 4.2 - Cont.

Last time:

$y'' - 3y' + 2y = 0$ Solution: $y = c_1 e^{t} + c_2 e^{2t}$ c_1, c_2 constant

e^t, e^{2t} comes from solution of $r^2 - 3r + 2 = 0$. $(r = 1, 2)$

Thm: Given $ay'' + by' + cy = 0$ where $a, b, c \in \mathbb{R}$ and $a \neq 0$
If the characteristic eqn.
$ar^2 + br + c = 0$ has two distinct, real roots, r_1 and r_2, then a general solution of the ODE is $y = c_1 e^{r_1 t} + c_2 e^{r_2 t}$, $c_1, c_2 \in \mathbb{R}$.

e.g. Find $y = y(t)$ s.t. $\begin{cases} y'' - 2y' - 3y = 0 \\ y(0) = 0, \ y'(0) = 1. \end{cases}$

$r^2 - 2r - 3 = 0$

$(r-3)(r+1) = 0$

$r_1, r_2 = 3, -1$

$\underbrace{y(t) = c_1 e^{3t} + c_2 e^{-t}}_{\text{Gen sol.}}$

$y(0) = 0$:

$0 = c_1 + c_2$

$c_1 = -c_2$

$y'(0) = 1$

$1 = 3c_1 e^{3t} + -c_2 e^{-t}$

$1 = 3c_1 - c_2$

$c_1 = 1/4, \ c_2 = -1/4$

$\boxed{y(t) = \frac{1}{4} e^{3t} - \frac{1}{4} e^{-t}}$

Question: What if our root has multiplicity >1?

e.g. $y'' = 0$

$r^2 = 0 \to r = 0$, double root (multiplicity = 2)

$e^{rt} = e^{0 \cdot t} = 1$ is a solution.
Can guess another: $y(t) = t$. Now we can have a 2-D solution space: $y(t) = c_1 \cdot \mathbb{1} + c_2 \cdot t$.

Check: $y = c_1 + c_2 t$ gives all sol to $y'' = 0$.
derivative of $y' = 0 \Rightarrow y'$ is constant.
$\Rightarrow y' = a_1$, a_1 is constant
$\Rightarrow y = \int y' dt = \int a_1 dt = \underline{a_1 t + a_2}$.

Thm $ay'' + by' + cy = 0$. If $ar^2 + br + c$ has a double root r_0 $(\Delta = b^2 - 4ac = 0,\ r_0 = -b/2a)$
Then, general solution is:
$y = c_1 e^{r_0 t} + c_1 t e^{r_0 t}$

Rmk Solution set is 2-dimensional with basis $e^{r_0 t}, te^{r_0 t}$
To-Do: Check $te^{r_0 t}$ is sol of ODE.

Wronskian
Question: Given f, g differentiable functions, determine whether they are linearly independent.
e.g. (1) $r_1 \neq r_2$, then $e^{r_1 t}, e^{r_2 t}$ are lin. ind.
(2) $e^{r_0 t}, te^{r_0 t}$ are lin. ind.
Thm f, g linearly independent iff $\det \begin{vmatrix} f(t) & g(t) \\ f'(t) & g'(t) \end{vmatrix} \neq 0$.

So check $f(t)g'(t) - f'(t)g(t) \neq 0$.
Useful: ① In theory.
② "numerically"

Thm (initial value theorem)

$$\begin{cases} ay'' + by' + c = 0 & a \neq 0 \\ y(t_0) = Y_0, \ y'(t_0) = Y_1 & Y_1, Y_0, t_0 \in \mathbb{R} \end{cases}$$

always has a unique solution.

Question: what if order > 2?

e.g. $y''' - y' = 0$

Sol $r^3 - r = 0 \Rightarrow r(r^2-1) = 0 \quad r = 0, -1, 1$

$y(t) = c_1 + c_2 e^t + c_3 e^{-t}$

e.g $y''' + y'' - y' - y = 0$

sol $r^3 + r^2 - r - 1 = 0$

$(r+1)(r^2-1) = 0$

$(r+1)^2(r-1) = 0 \rightarrow r_1 = -1$ - double

$r_2 = 1$ - single

Basis of sol. set: e^{-t}, e^t, te^{-t}.

because -1 is double root

$y(t) = c_1 e^{-t} + c_2 te^{-t} + c_3 e^t$.

e.g $y''' + 3y'' - 3y' + y = 0$

Sol $(r+1)^3 = 0$, $r = -1$ ← triple

Basis of sol set: $e^{-t}, te^{-t}, t^2 e^{-t}$.

$y(t) = c_1 e^{-t} + c_2 t e^{-t} + c_3 t^2 e^{-t}$.