§7.1 Diagonalization of Symmetric Matrices

Recall: A $n \times n$ matrix

eigenvalues = sol of $\det(A - \lambda I) = 0$

Thm A diagonalizable $\Leftrightarrow$ $\forall$ eigen value λ_i,
dim eigenspace (λ_i) = mult.

If this holds, eventually get $A = PDP^{-1}$

$$D = \begin{bmatrix} \lambda_1 & & \\ & \ddots & \\ & & \lambda_n \end{bmatrix} \qquad P = \begin{bmatrix} \vec{ev}_1 & \dots & \vec{ev}_n \end{bmatrix}$$

Rmk Not every square matrix is diagonalizable

Dfn A symmetric matrix is an $n \times n$ matrix satisfying
$A^T = A$.

e.g. $\begin{bmatrix} -1 & c & b \\ c & 2 & a \\ b & a & 5 \end{bmatrix}$

Thm Any symmetric matrix is diagonalizable.

e.g

$A = \begin{bmatrix} 1 & 2 \\ 2 & 1 \end{bmatrix}$ $\det(A - \lambda I) = 0 = (\lambda - 1)^2 - 2^2$

$(\lambda - 1)^2 = 4$

$\lambda = 3, -1$

$\lambda = -1$

$\begin{bmatrix} 2 & 2 \\ 2 & 2 \end{bmatrix}$ span$\left(\left\{\begin{bmatrix} 1 \\ -1 \end{bmatrix}\right)\right.$

$\lambda = 3$

$\begin{bmatrix} -2 & 2 \\ 2 & -2 \end{bmatrix}$ - span$\left(\begin{bmatrix} 1 \\ 1 \end{bmatrix}\right)$

✓ Diagonalizable

Observation $\begin{bmatrix} 1 \\ -1 \end{bmatrix}, \begin{bmatrix} 1 \\ 1 \end{bmatrix}$ are orthogonal.

Rescaling: $\frac{1}{\sqrt{2}}\begin{bmatrix} 1 \\ -1 \end{bmatrix}, \frac{1}{\sqrt{2}}\begin{bmatrix} 1 \\ 1 \end{bmatrix}$ → orthonormal basis

$A = PDP^{-1}$

$P = \begin{bmatrix} 1/\sqrt{2} & 1/\sqrt{2} \\ -1/\sqrt{2} & 1/\sqrt{2} \end{bmatrix}$ $D = \begin{bmatrix} -1 & 0 \\ 0 & 3 \end{bmatrix}$, P is orthogonal matrix

$P^TP = I = PP^T$, $P^{-1} = P^T$

Thm If A is symmetric, then A is diagonalizable by an orthogonal matrix.

(so if A is symmetric, then any 2 e.vectors of diffeent eigenvalues are orthogonal

→ $A\vec{v}_1 = \lambda_1\vec{v}_1$

$A\vec{v}_2 = \lambda_2\vec{v}_2$

Want to show $\vec{v}_1 \cdot \vec{v}_2 = 0$ (Compute $\vec{v}_2^{\,t} A\vec{v}_1$)

① $\vec{v}_2^{\,T}(A\vec{v}_1) = \vec{v}_2^{\,T}(\lambda_1\vec{v}_1) = \lambda_1(\vec{v}_2^{\,T}\vec{v}_1)$

② $(\vec{v}_2^{\,T}A)\vec{v}_1 = (Av_2)^T v_1 = \lambda_2(\vec{v}_2^{\,T}\vec{v}_1)$

(Note: $(A\vec{v}_2)^T = \vec{v}_2^{\,T}A^T = \vec{v}_2^{\,T}A$)

Algorithm to find orth. diagonalization for sym. A:

① Solve $\det(A - \lambda I) = 0$, get eigenvalues

② For each λ_i, find basis of eigensp.
$V_i = \text{nul}(A - \lambda_i I)$

③ Use Gram-Schmidt to find orthonormal basis of V_i.

④ Form P by orthonormal bases of $V_1, V_2, \ldots, V_s$

If A is symmetric,

$A = PDP^{-1} = PDP^T$

$A^T = (PDP^T)^T = (P^T)^T D^T P^T = PDP^T$ ✓