## §6.5 Least Square

If $A\vec{x} = \vec{b}$ is inconsistent (no solution), what can we do?

e.g Pigs: factors for the growth

- food $x_1$
- sleep $x_2$
- sports $x_3$
- temp. $x_4$
- water $x_5$

Goal: weight $y$.

"Expect": $y = a_1x_1 + a_2x_2 + \dots + a_5x_5$

Approx want to find $a_1, \dots, a_5$.

To find $a_1, \dots, a_5$,

for each pig, find $x_1, \dots, x_5$ and $y$. (eqn 1)

$x_1a_1 + x_2a_2 + \dots + x_5a_5 = y$

If you have 100 pigs, get 100 eqns

→ overdetermined! ($100 >> 5$)

But we still want the best approx.

Dfn A $m \times n$ matrix

$A\vec{x} = \vec{b}$, $\vec{x} \in \mathbb{R}^n$, $\vec{b} \in \mathbb{R}^m$ ($\vec{x}$ variable, $\vec{b}$ constant)

A least square solution is a vector $\hat{x} \in \mathbb{R}^n$ s.t. $\|A\hat{x} - b\|$ is minimal at $\hat{x}$.

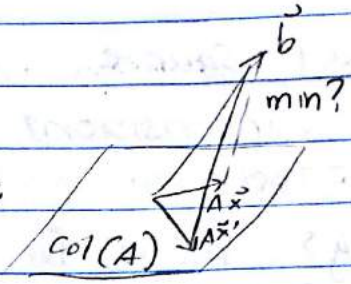
Rmk If eqn is consistent, the least square sol is the usual sol.

Idea $\min(\|A\vec{x} - \vec{b}\|)$

$= \text{dist}(\text{Col}(A), \vec{b})$

Smallest value is given by:

$\text{proj}_{\text{Col}(A)}(\vec{b})$.




Denote $\hat{b} = \text{proj}_{\text{Col}(A)}(\vec{b})$

Then, $\min(\|A\vec{x} - \vec{b}\|) = \|\hat{b} - \vec{b}\|$. Find $\vec{x}$ s.t. min achieved

$A\vec{x} = \hat{b}$ (consistent)

$\hat{b} - \vec{b} \in \text{Col}(A)^{\perp} = \text{nul}(A^T)$

$\Rightarrow A^T(\hat{b} - \vec{b}) = 0 \Rightarrow A^T(\hat{b}) = A^T(\vec{b})$

So $A\vec{x} = \hat{b}$ becomes $\boxed{A^TA\vec{x} = A^T\vec{b}}$

Rmk ① The eqn $A^TA\vec{x} = A^T\vec{b}$ is consistent.

② $A$ $m \times n$

$A^TA$ $n \times n$, symmetric $(A^TA)^T = A^TA$

Thm The least square sol. is unique

$\Leftrightarrow A^TA$ is invertible

$\Leftrightarrow$ columns of $A$ are linearly indep.

($\text{rank}(A) = n$)