§6.4 Gram-Schmidt process

03/14

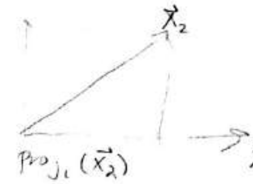
Process to compute orthogonal / orthonormal basis

$\Rightarrow$ Any subspace of $\mathbb{R}^n$ has an orthonormal basis

eg $W \subset \mathbb{R}^3$ subspace, spanned by $\vec{x}_1 = \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix}$, $\vec{x}_2 = \begin{bmatrix} 1 \\ -1 \\ 0 \end{bmatrix}$.

Find an orthogonal basis of W.

Sol $\vec{x}_1, \vec{x}_2$ are linearly independent (basis of W), but $\vec{x}_1 \cdot \vec{x}_2 = -1$, so it is not an orthogonal basis.



Take $L = \text{span}\{\vec{x}_1\}$

$L^{\perp}$: orthogonal component

Use the new basis:

$\vec{v}_1 = \vec{x}_1$

$\vec{v}_2 = proj_{L^{\perp}}(\vec{x}_2) = \vec{x}_2 - proj_L(\vec{x}_2)$

$$= \vec{x}_2 - \frac{\vec{x}_2 \cdot \vec{x}_1}{\vec{x}_1 \cdot \vec{x}_1}\vec{x}_1 = \begin{bmatrix} 1 \\ -1 \\ 0 \end{bmatrix} - \frac{-1}{14}\begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix} = \begin{bmatrix} 15/14 \\ -6/7 \\ 3/14 \end{bmatrix}$$

Rmk: To get an orthonormal basis, compute $\frac{\vec{v}_1}{\|\vec{v}_1\|}, \frac{\vec{v}_2}{\|\vec{v}_2\|}$.

$$= \begin{bmatrix} \frac{1}{\sqrt{14}} \\ \frac{2}{\sqrt{14}} \\ \frac{3}{\sqrt{14}} \end{bmatrix}, \sqrt{\frac{27}{14}}\begin{bmatrix} 15/14 \\ -6/7 \\ 3/14 \end{bmatrix}$$

Thm Gram-Schmidt

$W \subset \mathbb{R}^n$, $\{\vec{x}_1, \dots, \vec{x}_p\}$ basis of W. An orthogonal basis of W is

$\vec{v}_1 = \vec{x}_1$

$\vec{v}_2 = \vec{x}_2 - \frac{\vec{x}_2 \cdot \vec{v}_1}{\vec{v}_1 \cdot \vec{v}_1}\vec{v}_1$

$\vec{v}_3 = \vec{x}_3 - \frac{\vec{x}_3 \cdot \vec{v}_1}{\vec{v}_1 \cdot \vec{v}_1}\vec{v}_1 - \frac{\vec{x}_3 \cdot \vec{v}_2}{\vec{v}_2 \cdot \vec{v}_2}\vec{v}_2 \quad \left(\vec{v}_3 = \vec{x}_3 - proj_{\text{span}(\vec{v}_1, \vec{v}_2)}(\vec{x}_3)\right)$

so $\vec{v}_p = \vec{x}_p - \frac{x_p \cdot \vec{v}_1}{\vec{v}_1 \cdot \vec{v}_1} \vec{v}_1 + \cdots +$

$= \text{proj}_{\text{span}(\{\vec{v}_1, \ldots, \vec{v}_{p-1}\})^\perp}(\vec{v}_p)$.

Moreover, $\vec{v}_1, \ldots, \vec{v}_p$ is orthog. basis of $\text{span}\{\vec{x}_1, \ldots, \vec{x}_i\}$, for $i = 1, 2, \ldots p$.

e.g. $i=3$, $p=100$ gives $\vec{v}_1, \vec{v}_2, \vec{v}_3$ is orth. basis of $\text{span}\{\vec{x}_1, \vec{x}_2, \vec{x}_3\}$.

<u>QR Factorization</u>

<u>Thm</u> A $m \times n$ matrix, with lin. indep. col (rank (A) $= n \leq m$)

Then we write $QR = A$.

where

① Q $m \times n$ matrix whose columns form an orthonormal basis of $\text{col}(A) = \text{col}(Q)$.

② R is $n \times n$ upper triangular matrix.

<u>Rmk</u> If A square matrix ($m=n$)

$A = QR$ = orthogonal matrix · upper-triangular matrix

<u>Reason</u> $A = [\vec{x}_1, \ldots, \vec{x}_p]$

Use GramSchmidt on $\{\vec{x}_1, \ldots, \vec{x}_p\}$

$\Rightarrow \vec{v}_1, \ldots, \vec{v}_p$ (orthonormal basis)

Then, $A = QR$, $Q = [\vec{v}_1, \ldots, \vec{v}_p]$

R = coefficient matrix for the relation from $\{\vec{x}_1, \ldots, \vec{x}_p\}$ to $\{\vec{v}_1, \ldots, \vec{v}_p\}$

<u>eg</u>

$A = \begin{bmatrix} 0 & 1 \\ 1 & 2 \\ 0 & 3 \end{bmatrix} = QR$?

Gram-schmidt

$v_1 = \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}$

$\vec{v}_2 = \frac{1}{\sqrt{10}} \begin{bmatrix} 1 \\ 0 \\ 3 \end{bmatrix}$

$\vec{v}_1 = 1\vec{x}_1$

$\vec{v}_2 = \frac{1}{\sqrt{10}}(\vec{x}_2 - 2\vec{v}_1) = \frac{1}{\sqrt{10}}(\vec{x}_2 - 2\vec{x}_1)$

$Q = \begin{bmatrix} 0 & 1 \\ 1 & 0 \\ 0 & 3 \end{bmatrix}$

$R = \begin{bmatrix} 1 & -2/\sqrt{10} \\ 0 & 1/\sqrt{10} \end{bmatrix}$