03/11

Orthogonal Matrix

① Columns are <u>orthonormal</u>

② Square matrix

<u>Thm</u> U $m \times n$ matrix

then: columns of U orthonormal $\Leftrightarrow U^T U = I_n$

<u>Reason</u>: Write $U = [\vec{u}_1 \dots \vec{u}_n]$

Then $U^T U = \begin{bmatrix} \vec{u}_1 \\ \vdots \\ \vec{u}_n \end{bmatrix} [\vec{u}_1 \dots \vec{u}_n] = \begin{bmatrix} 1 & & 0 \\ & \ddots & \\ 0 & & 1 \end{bmatrix}$

<u>Thm</u> U $m \times n$ orthonormal columns, then

$\|U\vec{x}\| = \|\vec{x}\|, \ \forall x \in \mathbb{R}^n$

$(U\vec{x}) \cdot (U\vec{y}) = \vec{x} \cdot \vec{y}$

($U : \mathbb{R}^n \to \mathbb{R}^m$, presences length)

<u>Proof</u> $\|U\vec{x}\|^2$

$= (U\vec{x}) \cdot (U\vec{x})$

$= (U\vec{x})^T (U\vec{x})$

$= x^T \underbrace{U^T U}_{I} \vec{x} = \|x\|^2.$

Thm U $n \times n$ square matrix
U orthogonal matrix $\Leftrightarrow$ U has orthonormal rows

Reason orthon. col : $U^T U = I_n$ $\Rightarrow U^T = U^{-1}$
orthon. row : $U U^T = I_n$

Warning: If U $m \times n$, $m \neq n$, and has orthonormal columns, U is not orthogonal matrix.
$U^T U = I_n$, but $U U^T \neq I_n$.

§6.3 Orthogonal Projection

Def $W \subset \mathbb{R}^n$ subspace
$W^\perp \subset \mathbb{R}^n$ orthogonal complement
$\vec{y} \in \mathbb{R}^n$ any vector - say $\vec{y} = \vec{y}_1 + \vec{y}_2$

$\vec{y}_1 \in W$, $\vec{y}_2 \in W^\perp$
Then, define proj (projection):
$\text{proj}_W(\vec{y}) = \vec{y}_1$
$\text{proj}_{W^\perp}(\vec{y}) = \vec{y}_2$

e.g

$$\vec{y} = \underbrace{\begin{bmatrix} a \\ b \\ 0 \end{bmatrix}}_{\text{proj}_W(\vec{y})} + \underbrace{\begin{bmatrix} 0 \\ 0 \\ c \end{bmatrix}}_{\text{proj}_{W^\perp}(\vec{y})}$$

Thm $W \subseteq \mathbb{R}^n$ subspace

$\vec{y} \in \mathbb{R}^n$

If W has an orthogonal basis,

$\{\vec{u}_1, \ldots, \vec{u}_p\}$

then, $proj_W(\vec{y}) = \left(\frac{\vec{y}_1 \cdot \vec{u}_1}{\vec{u}_1 \cdot \vec{u}_1}\right)\vec{u}_1 + \ldots + \left(\frac{\vec{y}_p \cdot \vec{u}_p}{\vec{u}_p \cdot \vec{u}_p}\right)\vec{u}_p$

$$proj_{W^\perp}(\vec{y}) = \vec{y} - proj_W(\vec{y})$$

case $\dim W = 1$, $W = span(\{\vec{w}\})$

$$proj_W(\vec{y}) = \left(\frac{\vec{y} \cdot \vec{w}}{\vec{w} \cdot \vec{w}}\right)\vec{w}.$$

Thm best approximation.

$W \subseteq \mathbb{R}^n$ subspace, $\vec{y} \in \mathbb{R}^n$

Then $proj_W(\vec{y})$ is the vector in W closet to $\vec{y}$.

$dist(\vec{y}, \vec{w}) = \|\vec{y} - \vec{w}\|$, $\vec{w} \in W$ has a min value at $\vec{w} = proj_W(\vec{y})$.

Thm $W \subseteq \mathbb{R}^n$, $\vec{y} \in \mathbb{R}^n$

$\{\vec{u}_1, \ldots, \vec{u}_p\}$ orthonormal basis

Then $proj_W(\vec{y}) = UU^T\vec{y}$ where $U = [\vec{u}_1, \ldots \vec{u}_p]$

Proof $proj_W(\vec{y}) = (y \cdot \vec{u}_1)\vec{u}_1 + \ldots + (y \cdot \vec{u}_p)\vec{u}_p$

$$UU^T\vec{y} = [\vec{u}_1 \ldots \vec{u}_p]\begin{bmatrix} \vec{u}_1 \\ \vdots \\ \vec{u}_p \end{bmatrix}\vec{y}$$

$$= (\vec{y} \cdot \vec{u}_1)\vec{u}_1 + \ldots + (\vec{y} \cdot \vec{u}_p)\vec{u}_p$$