03/09

Recall: Orthogonal complement

W subspace of $\mathbb{R}^n$

$W^\perp = \{\vec{x} \in \mathbb{R}^n \mid x \perp W\}$

FACTS:

① $\dim W + \dim W^\perp = n$

② $W^\perp = H \Leftrightarrow H^\perp = W$

e.g. (1) $(\text{line in } \mathbb{R}^3)^\perp = (\text{plane in } \mathbb{R}^3)$

(2) $(\text{plane in } \mathbb{R}^3)^\perp = (\text{line in } \mathbb{R}^3$

(3) $W = \mathbb{R}^n$ in $\mathbb{R}^n$, $W^\perp = 0 = \{\vec{0}\}$

How to compute orthogonal complement $W \subset \mathbb{R}^n$

(1) find spanning set $\{\vec{v}_1, \dots, \vec{v}_p\}$ of W.

(2) solve eqn:

$$\vec{v}_1 \cdot \vec{x} = 0, \dots, \vec{v}_p \cdot \vec{x} = \vec{0}$$

$$\vec{v}_1^T \vec{x} = 0, \dots, \vec{v}_p^T \vec{x} = \vec{0}$$

Rewrite as: $\begin{bmatrix} \vec{v}_1^T \\ \vdots \\ \vec{v}_p^T \end{bmatrix} \vec{x} = \vec{0}$

This is just $A^T \vec{x} = 0$.

$A = [\vec{v}_1 \dots \vec{v}_p]$

Conclusion $W^\perp = \mathrm{nul}(A^T)$

Thm: A matrix $n \times p$, then

① $(\mathrm{col}(A))^\perp = \mathrm{nul}(A^T)$

② $(\mathrm{row}(A))^\perp = \mathrm{nul}(A)$

§ 6.2 Orthogonal sets

$S = \{\vec{u}_1, \dots, \vec{u}_p\}$ set in $\mathbb{R}^n$

① S is an orthogonal set if any two vectors of S are orthogonal to each other

② S is an orthonormal set if

ⓐ each pair of distinct vec. of S are orthogonal

ⓑ all vector of S has length 1.

$(\vec{u}_i \cdot \vec{u}_j = 0 \ \forall i \neq j \ ; \ \|\vec{u}_i\| = 1, \ \forall i)$

Thm An orthogonal set of nonzero vectors is linearly independent.

Proof: Say $c_1\vec{u}_1 + \dots + c_p\vec{u}_p = \vec{0}$

want: $c_1, \dots, c_p = 0$. (dfn of lin. ind.)

Consider:

$$\vec{u}_1 \cdot (c_1\vec{u}_1 + c_2\vec{u}_2 + \dots + c_p\vec{u}_p)$$

$$= c_1 \vec{u}_1 \cdot \vec{u}_1 + 0 + 0 + \dots + 0$$

Then $c_1 = 0$. (and $c_2, \dots, c_p = 0$)

Dfn An orthogonal basis of $\mathbb{R}^n$ is a basis which is also an orthogonal set.

Dfn An orthonormal basis of $\mathbb{R}^n$ is a basis which is also an orthonormal set.

e.g. Standard basis: $\{\vec{e}_1, \dots, \vec{e}_n\}$ is orthonormal

e.g in $\mathbb{R}^2$, the basis $\begin{bmatrix} \cos\theta \\ \sin\theta \end{bmatrix}, \begin{bmatrix} -\sin\theta \\ \cos\theta \end{bmatrix}$ is orthonormal.

e.g. $\vec{u}_1 \begin{bmatrix} 3 \\ 1 \\ 1 \end{bmatrix}$, $\vec{u}_2 = \begin{bmatrix} -1 \\ 2 \\ 1 \end{bmatrix}$, $\vec{u}_3 = \begin{bmatrix} -1 \\ -4 \\ 7 \end{bmatrix}$

orthogonal basis of $\mathbb{R}^3$.

* If you have an orthogonal basis, you don't need to check lin. ind.

Thm If $\{\vec{u}_1, \ldots, \vec{u}_p\}$ orthogonal basis of W in $\mathbb{R}^n$, then any $\vec{y} \in W$,

$$\vec{y} = \frac{\vec{u}_1 \cdot \vec{y}}{\vec{u}_1 \cdot \vec{u}_1} \vec{u}_1 + \ldots + \frac{\vec{u}_p \cdot \vec{y}}{\vec{u}_p \cdot \vec{u}_p} \vec{u}_p$$

$$\vec{y} = c_1 \vec{u}_1 + \ldots + c_p \vec{u}_p$$

$$c_i = \frac{\vec{u}_i \cdot \vec{y}}{\vec{u}_i \cdot \vec{u}_i}$$

Reason: $\vec{y} \cdot \vec{u}_1 = (c_1 \vec{u}_1 + \ldots + c_p \vec{u}_p) \cdot \vec{u}_1 = c_1 \vec{u}_1 \cdot \vec{u}_1$

so $c_1 = \dfrac{\vec{y} \cdot \vec{u}_1}{\vec{u}_1 \cdot \vec{u}_1}$

e.g. $\vec{u}_1 = \begin{bmatrix} 3 \\ 1 \\ 1 \end{bmatrix}$ $\vec{u}_2 = \begin{bmatrix} -1 \\ 2 \\ 1 \end{bmatrix}$, $\vec{u}_3 = \begin{bmatrix} -1 \\ -4 \\ 7 \end{bmatrix}$

write $y = \begin{bmatrix} 6 \\ 1 \\ 8 \end{bmatrix}$ as lin. combo of bases

Sol: Apply

$$c_1 = \frac{\vec{u}_1 \cdot \vec{y}}{\vec{u}_1 \cdot \vec{u}_1}$$ woo!

$$c_2 = \frac{\vec{u}_2 \cdot \vec{y}}{\vec{u}_2 \cdot u_2}$$

$\vdots$

$c_3 = \cdots$

Dfn An orthogonal matrix is an $n \times n$ matrix whose columns form an orthonormal basis of $\mathbb{R}^n$.

Rmk: Linear trans. of Orthogonal matrix is a composition of rotation + reflection.