Midterm 2: 03/16 - covers sections after MT1
CHEATSHEET OK!

§6.1 Inner Product (also called dot product)

Dfn $\vec{u} = \begin{bmatrix} u_1 \\ \vdots \\ u_n \end{bmatrix}$ $\vec{v} = \begin{bmatrix} v_1 \\ \vdots \\ v_n \end{bmatrix}$ vectors in $\mathbb{R}^n$.

The inner product is:

$$\vec{u} \cdot \vec{v} = \begin{bmatrix} u_1 v_1 \\ \vdots \\ u_n v_n \end{bmatrix}$$

other notations: $\vec{u}^T \vec{v}$, $\vec{v}^T \vec{u}$ ← like matrix mult.

Properties:

1. $\vec{u} \cdot \vec{v} = \vec{v} \cdot \vec{u}$
2. $\vec{u}(\vec{v} + \vec{w}) = \vec{u} \cdot \vec{v} + \vec{u} \cdot \vec{w}$
3. $(c\,\vec{u}) \cdot \vec{v} = \vec{u}(c\,\vec{v}) = c(\vec{u} \cdot \vec{v})$
4. $\vec{u} \cdot \vec{u} \geq 0$ and $\vec{u} \cdot \vec{u} = 0 \Leftrightarrow \vec{u} = \vec{0}$
   $\rightarrow \vec{u} \cdot \vec{u} = u_1^2 + u_2^2 + u_3^2 + \ldots + u_n^2 \geq 0$.

Dfn The length (norm) of a vector $\vec{u}$ is

$$\|\vec{u}\| = \sqrt{\vec{u} \cdot \vec{u}} = \sqrt{u_1^2 + u_2^2 + \ldots + u_n^2}$$

magnitude

Dfn $\vec{u}$ is called a unit vector if $\|\vec{u}\| = 1$

ex. $u = \begin{bmatrix} -1 \\ 2 \\ 3 \end{bmatrix} \in \mathbb{R}^3$ → find unit vector in direction of u.

→ want to find c s.t. $\|c\vec{u}\| = 1$

$\|c\vec{u}\| = 1$

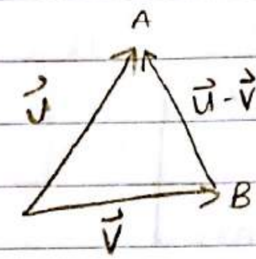
$= c\|\vec{u}\| = 1 \quad \longrightarrow \quad c = \dfrac{1}{\|\vec{u}\|}$

The unit vector

$$\frac{1}{\|\vec{u}\|}\vec{u} = \frac{1}{\sqrt{(-1)^2+2^2+3^2}}\vec{u} = \boxed{\frac{1}{\sqrt{14}}\begin{bmatrix}-1\\2\\3\end{bmatrix}}$$

<u>Dfn</u> $\vec{u}, \vec{v} \in \mathbb{R}^n$

<u>$\text{dist}(\vec{u}, \vec{v}) = \|\vec{u} - \vec{v}\|$</u>.

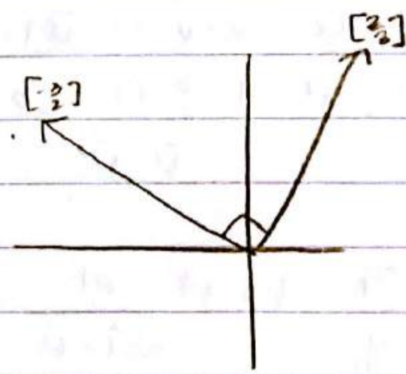


<u>Intuition</u>: distance of A,B is is length of $\overrightarrow{BA}$

$$\text{dist}(\vec{u}, \vec{v}) = \sqrt{(u_1-v_1)^2 + (u_2-v_2)^2 + \quad + (u_n - v_n)^2}$$

<u>Dfn</u> $\vec{u}, \vec{v} \in \mathbb{R}^n$. $\vec{u}, \vec{v}$ are <u>orthogonal</u> (perpendicular) if $\vec{u} \cdot \vec{v} = 0$.

<u>e.g</u> 2-dim case: $\begin{bmatrix}2\\3\end{bmatrix} \perp \begin{bmatrix}-3\\2\end{bmatrix}$.



<u>Thm</u> $\vec{u}, \vec{v}$ are orthogonal $\Leftrightarrow \|\vec{u}+\vec{v}\|^2 = \|u\|^2 + \|v\|^2$

- intuition comes from Pythagorean Thm.

<u>Pf</u> $\|\vec{u}+\vec{v}\| = (\vec{u}+\vec{v})\cdot(\vec{u}+\vec{v})$

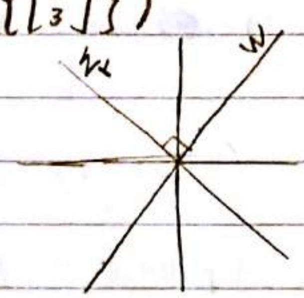
$$= \vec{u}\cdot\vec{u} + \underbrace{2\vec{u}\cdot\vec{v}}_{0 \text{ if orthogonal}} + \vec{v}\cdot\vec{v}$$

$$= \|\vec{u}\|^2 + \|\vec{v}\|^2$$

Dfn $W \subset \mathbb{R}^n$ is a subspace.
A vector $\vec{z} \in \mathbb{R}^n$ is orthogonal to W if $\vec{z}$ is orthogonal every vector in W.
The orthogonal complement of $W \subset \mathbb{R}^n$ is the set of all vectors of $\mathbb{R}^n$ orthogonal to W.

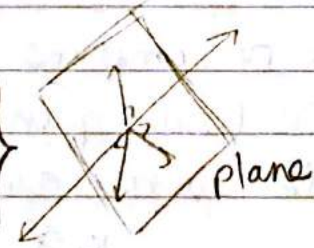
e.g. ① in $\mathbb{R}^2$, $W = \operatorname{span}\left(\left\{\begin{bmatrix} 2 \\ 3 \end{bmatrix}\right\}\right)$

$$W^{\perp} = \operatorname{span}\left(\left\{\begin{bmatrix} -3 \\ 2 \end{bmatrix}\right\}\right)$$



② In $\mathbb{R}^3$, $W = \operatorname{span}\left(\left\{\begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix}\right\}\right)$ ← "line" in $\mathbb{R}^3$

$$W^T = \left\{\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} \middle| \, x_1 + 2x_2 + 3x_3 = 0\right\}$$



↑
solve system.

get $\operatorname{span}\left(\left\{\begin{bmatrix} -2 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} -3 \\ 0 \\ 1 \end{bmatrix}\right\}\right)$
(2-dim)