03/04

**FACTS** A $n \times n$ matrix, $\det(A - \lambda I) = 0$ ← Polynomial of deg **n**

① If it has $n$ (pairwise) (no repeats) distinct real roots, then diagonalizable (all eigenspace are 1-dimensional)

② If it has imaginary roots, its not diagonalizable (because we are considering matrices of $\mathbb{R}$)

③ If all roots are real (but with multiplicity), then it depends.

In ③, diagonalizable iff $\forall$ root $\lambda_i$,
dim(eigenspace $\lambda_i$) = multiplicity of $\lambda_i$ as root.

e.g. ① $A = \begin{bmatrix} 0 & 2 \\ -3 & 0 \end{bmatrix} \rightarrow \lambda^2 = -6$
→ not diagonalizable

② $A = \begin{bmatrix} 0 & 2 \\ 3 & 0 \end{bmatrix} \rightarrow \lambda^2 = 6$
→ diagonalizable

③ $A = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix} \rightarrow \lambda^2 = 0$ w/ multiplicity 2
(will find that not diagonaliza)

④ $A = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 5 \\ 0 & 0 & 2 \end{bmatrix} \rightarrow \lambda^2(2 - \lambda) = 0$
$\lambda = 0$ w/ mult 2
$\lambda = 2$ w/ mult. 1
(will find that is diagoneliz)

$\lambda = 0$,
get span$\left(\left\{\begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}\right\}\right)$

$\lambda = 2$,
get span$\left(\left\{\begin{bmatrix} 0 \\ 5/2 \\ 1 \end{bmatrix}\right\}\right)$

$A = PDP^{-1}$

$D = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 2 \end{bmatrix}$

$P = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 5/2 \\ 0 & 0 & 1 \end{bmatrix}$

Rmk: ① Choice of eigenvectors not unique

e.g. $\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 5 \\ 0 & 0 & 2 \end{bmatrix}$ OK

$\begin{bmatrix} 0 & -1 & 0 \\ -1 & 0 & 5 \\ 0 & 0 & 2 \end{bmatrix}$ OK

② order of eigenvalues not unique

e.g. $D = \begin{bmatrix} 2 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$ $P = \begin{bmatrix} 0 & 1 & 0 \\ 5/2 & 0 & 1 \\ 0 & 0 & 0 \end{bmatrix}$

order of eigenvalues same as order of eigenvectors

§5.4 Linear Trans

Def: $T: V \to W$ lin trans

$\mathcal{B} = \{\vec{b}_1, \ldots, \vec{b}_n\}$ basis of $V$

$\mathcal{C} = \{\vec{c}_1, \ldots, \vec{c}_n\}$ basis of $W$

The matrix $A$ that represents $T$ relative to $\mathcal{B}, \mathcal{C}$ is the matrix representing the transformation $T': \mathbb{R}^m \to \mathbb{R}^n$ on the coordinates w/ respect to $\mathcal{B}$ and $\mathcal{C}$

$$\begin{tikzcd} V \arrow[r, "T"] \arrow[d, "\text{coord map}"'] & W \arrow[d, "\text{coord map}"] \\ \mathbb{R}^n \arrow[r, "T'"] & \mathbb{R}^m \end{tikzcd} \qquad \begin{tikzcd} \vec{w} \arrow[d] \\ {[\vec{w}]_c} \end{tikzcd}$$

e.g. $T : V \to W$

$\mathcal{B} = \{\vec{b}_1, \vec{b}_2, \vec{b}_3\}$ $\dim(V)=3$

$\mathcal{C} = \{\vec{c}_1, \vec{c}_2\}$ $\dim(W)=2$

$T(\vec{b}_1) = \vec{c}_1 + \vec{c}_2$ | $T\left(\begin{bmatrix}1\\0\\0\end{bmatrix}\right) = \begin{bmatrix}1\\1\end{bmatrix}$

$T(\vec{b}_2) = \vec{c}_1 - \vec{c}_2$ | $T\left(\begin{bmatrix}0\\1\\0\end{bmatrix}\right) = \begin{bmatrix}1\\-1\end{bmatrix}$

$T(\vec{b}_3) = 5\vec{c}_2$ | $T\left(\begin{bmatrix}0\\0\\1\end{bmatrix}\right) = \begin{bmatrix}0\\5\end{bmatrix}$

The matrix $= \begin{bmatrix} 1 & 1 & 0 \\ 1 & -1 & 5 \end{bmatrix}$

Thm $A$ $n\times n$, diagonalizable

$A = PDP^{-1}$

$P = [\vec{v}_1 \dots \vec{v}_n]$, $D = \begin{bmatrix} \lambda_1 & & 0 \\ & \ddots & \\ 0 & & \lambda_n \end{bmatrix}$

The linear transformation, $T(x) = Ax$ is represented by $D$ relative the basis $\{\vec{v}_1, \dots, \vec{v}_n\}$

Reason. $T\vec{v}_i = \lambda_i \vec{v}_i$

$T\begin{pmatrix} 0 \\ 1 \\ \vdots \\ 0 \end{pmatrix} = \begin{bmatrix} 0 \\ \lambda_2 \\ \vdots \\ 0 \end{bmatrix}$

$\det(A - \lambda I) =$
$= \det(PBP^{-1} - \lambda I)$
$= \det(PBP^{-1} - \lambda P I P^{-1})$
$= \det(P(BP^{-1} - \lambda I P^{-1}))$
$= \det(P(B - \lambda I)P^{-1})$

Dfn $A, B$ $n\times n$ matrices, Say $A$ is similar to $B$ if $A = PBP^{-1}$ for some inv. matrix $P$

(P does not have to be diagonal)

Rmk Diagonalizable means similar to diagonal matrix

Also, $A = PBP^{-1} \Leftrightarrow B = P^{-1}AP$, $A$ + $B$ have same eigenvalues