03/03

Inner Product Space:
A vector space where you can take dot products.

Dfn $T: V \to V$
$\lambda \in \mathbb{C}$

λ is an e.v. of T iff $\exists\ v_\lambda \in V,\ v_\lambda \neq 0$ s.t
$T(v_\lambda) = \lambda v_\lambda$ $\quad v_\lambda$ is an evc (eigenvector)
$T(0) = \lambda 0$
ev can be 0, v_λ cannot.

$V := \{v \in V : T(v) = \lambda v\}$ is e. space

$T(v) = \lambda v,\ T(w) = \lambda w \Rightarrow T(v+w) = \lambda(v+w)$
$\Downarrow$
$T(\alpha v) = \lambda(\alpha v)$
~~but not a vector space (no $\vec{0}$)~~

$V_\lambda = \{\text{evc w/ ev } \lambda\} \cup \{0\}$

Thm $T: V \to V$ lin. trans
$\dim(V) < \infty$
$\lambda \in \mathbb{C}$ is an ev. iff $\det(T - \lambda \, id_V) = 0$

$\chi_T(\lambda) := \det(T - \lambda \, id_V) \rightarrow$ characteristic polynomial

Prf λ is e.v. iff $\exists\ v_\lambda \in V, v_\lambda \neq 0$ w/ $T(v_\lambda) = \lambda v_\lambda$ iff $(T - \lambda\, id_V)(v_\lambda) = 0$ iff $\ker(T - \lambda\, id_V) \neq 0$ iff $T - \lambda\, id_V$ is not injective iff $T - \lambda\, id_V$ is not bijective iff $T - \lambda\, id_V$ is not invertible iff $\det(T - \lambda\, id_V) = 0$

$$A = \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix}$$

$$A - \lambda\, id_V = \begin{bmatrix} -\lambda & -1 \\ 1 & -\lambda \end{bmatrix}$$

$$\chi_A(\lambda) := \det(A - \lambda) = -\lambda^2 + 1$$

e v: $\lambda = \pm i$

$$V_\lambda = \ker(T - \lambda)$$

$$A - \lambda_+ = \begin{bmatrix} -i & -1 \\ 1 & -i \end{bmatrix}$$

$$Nul(A - \lambda_+) = span\left(\left\{\begin{bmatrix} 1 \\ -i \end{bmatrix}\right\}\right) \quad \left(\begin{bmatrix} -1 \\ i \end{bmatrix}, \begin{bmatrix} i \\ 1 \end{bmatrix}\right)$$

$$V_{+\lambda} = \begin{bmatrix} i & -1 \\ 1 & i \end{bmatrix} \qquad \begin{matrix} -i\,(1 & i) \\ -i & 1 \end{matrix}$$

$$\sim \begin{bmatrix} 0 & 0 \\ 1 & i \end{bmatrix} = \begin{bmatrix} -i \\ 1 \end{bmatrix}$$

$$A = \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix}$$

$$\begin{bmatrix} 1-\lambda & 1 \\ 0 & 1-\lambda \end{bmatrix} \qquad \begin{aligned} \lambda^2 - 2\lambda + 1 &= 0 \\ (1-\lambda)^2 &= 0 \\ \lambda &= 1. \end{aligned}$$

$\lambda = 1$,

$$\begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix} = \operatorname{span}\left(\left\{\begin{bmatrix} 1 \\ 0 \end{bmatrix}\right\}\right)$$

ev. uppertriangular matrix are diagonal vals.

A matrix is diagonalizable iff dim(e.s.) = mult.

<u>Dfn</u> $T: V \to V$ is diagonalizable iff $\exists\ \mathcal{B}$ basis s.t. $[T]_{\mathcal{B}} := [T]_{\mathcal{B} \leftarrow \mathcal{B}}$ is diagonal
↳shorthand for

<u>Prp</u> $[T]_{\mathcal{B}}$ is diagonal iff each $\vec{b}$ in $\mathcal{B}$ is an eigenvector of T.

Pf: $T(b) = \lambda b$

$\mathcal{B} = \{\vec{b}_1, \ldots, \vec{b}_d\}$. $[T(b_k)] = \begin{bmatrix} 0 \\ \vdots \\ \lambda \\ \vdots \\ 0 \end{bmatrix}$ ← k^{th} component

$[T]_{\mathcal{B}} = [[T(b_1)]_{\mathcal{B}} \ldots [T(b_d)]_{\mathcal{B}}]$

kth col of that matrix

(Cor: T is diagonalizable iff V has a basis of evc of T)

<u>Thm</u> T is diagonalizable iff $\forall$ ev λ,
$\dim(V_\lambda)$ = multiplicity of λ

<u>Thm</u> A $d \times d$ matrix $T_A : \mathbb{R}^d \to \mathbb{R}^d$, $T_A(x) = Ax$

T_A is diagonalizable so $\exists$ basis $\mathcal{B}$ of e vc
$\mathcal{B} = \{b_1, \ldots, b_d\}$. Then,

(i) $D := [T_A]_{\mathcal{B}} = \begin{bmatrix} \lambda & & 0 \\ & \ddots & \\ 0 & & \lambda_d \end{bmatrix}$

(ii) $D = P^{-1} A P$ $\qquad P = [b_1, \ldots, b_k]$
$= [id]_{\mathcal{E} \leftarrow \mathcal{B}}$
standard

$[T_A]_{\mathcal{B} \leftarrow \mathcal{B}} = [id]_{\mathcal{B} \leftarrow \mathcal{E}} [T_A]_{\mathcal{E} \leftarrow \mathcal{E}} [id]_{\mathcal{E} \leftarrow \mathcal{B}}$