03/02

§5.1, 2a(c) removed

§5.2 Characteristic Equation
§5.5 Diagonalization

Recall: A $n \times n$.

if $A\vec{x} = \lambda\vec{x}$ $\quad \vec{x} \in \mathbb{R}^n \quad \vec{x} \neq \vec{0}$

$\lambda \in \mathbb{R}$

Say $\vec{x}$ eigenvector

λ eigenvalue

Solution set of $A\vec{x} = \lambda\vec{x}$ (for fixed λ)
called eigenspace associated w/ λ.

$$A\vec{x} = (\lambda I)\vec{x}$$

$(A - \lambda I)\vec{x} = \vec{0}$ (has non-zero sol. $\vec{x}$)

so $\det(A - \lambda I) = 0$.

e.g $A = \begin{bmatrix} 1 & 6 \\ 5 & 2 \end{bmatrix}$ find all eigenvalues and eigenvectors

step 1: $\det(A - \lambda I)$

$$= \det\left(\begin{bmatrix} 1 & 6 \\ 5 & 2 \end{bmatrix} - \begin{bmatrix} \lambda & 0 \\ 0 & \lambda \end{bmatrix}\right)$$

$$= \det\left(\begin{bmatrix} 1-\lambda & 6 \\ 5 & 2-\lambda \end{bmatrix}\right) = 0$$

$$= (1-\lambda)(2-\lambda) - 30 = 0$$

$$2 + \lambda^2 - 3\lambda - 30 = 0$$

$$\lambda^2 - 3\lambda - 28 = 0$$

$(\lambda - 7)(\lambda + 4) = 0 \rightarrow \lambda = 7, -4$

(eigenvalues)

Step 2: Find eigenvectors.

$\lambda = 7$

$(A - 7I)\vec{x} = 0$

$$\begin{bmatrix} -6 & 6 \\ 5 & -5 \end{bmatrix}$$

$$\sim \begin{bmatrix} -1 & 1 \\ 1 & -1 \end{bmatrix}$$

$$\sim \begin{bmatrix} -1 & 1 \\ 0 & 0 \end{bmatrix} \quad \begin{aligned} x_1 &= x_2 \\ x_2 &= x_2 \end{aligned} \quad \text{Sol: } \boxed{\operatorname{span}\left(\left\{\begin{bmatrix} 1 \\ 1 \end{bmatrix}\right\}\right)}$$

$\lambda = -4$

$$\begin{bmatrix} 5 & 6 \\ 5 & 6 \end{bmatrix}$$

$$\sim \begin{bmatrix} 5 & 6 \\ 0 & 0 \end{bmatrix} \quad \begin{aligned} x_1 &= -6/5\, x_2 \\ x_2 &= x_2 \end{aligned} \quad \text{Sol: } \boxed{\operatorname{span}\left(\left\{\begin{bmatrix} -6/5 \\ 1 \end{bmatrix}\right\}\right)}$$

Diagonalization:

$$A = \begin{bmatrix} 1 & 6 \\ 5 & 2 \end{bmatrix}$$

Calculation gives:

$$A \begin{bmatrix} 1 \\ 1 \end{bmatrix} = 7 \begin{bmatrix} 1 \\ 1 \end{bmatrix}$$

$$A \begin{bmatrix} -6 \\ 5 \end{bmatrix} = -4 \begin{bmatrix} -6 \\ 5 \end{bmatrix}$$

Combine:

$$A \begin{bmatrix} 1 & -6 \\ 1 & 5 \end{bmatrix} = \begin{bmatrix} 1 & -6 \\ 1 & 5 \end{bmatrix} \begin{bmatrix} 7 & 0 \\ 0 & -4 \end{bmatrix}$$

$$\left(A\begin{bmatrix} 1 \\ 1 \end{bmatrix} A\begin{bmatrix} -6 \\ 5 \end{bmatrix}\right) = \left(7\begin{bmatrix} 1 \\ 1 \end{bmatrix} \ -4\begin{bmatrix} -6 \\ 5 \end{bmatrix}\right)$$

Multiply $\begin{bmatrix} 1 & -6 \\ 1 & 5 \end{bmatrix}^{-1}$ to the right.

Get: $A = \underbrace{\begin{bmatrix} 1 & -6 \\ 1 & 5 \end{bmatrix} \underset{\substack{\uparrow \\ \text{diagonal matrix}}}{\begin{bmatrix} 7 & 0 \\ 0 & -4 \end{bmatrix}} \begin{bmatrix} 1 & -6 \\ 1 & 5 \end{bmatrix}^{-1}}_{\text{diagonal form}}$

ex. compute A^n for $A = \begin{bmatrix} 1 & 6 \\ 5 & 2 \end{bmatrix}$

$A = PDP^{-1}$ $\quad D = \begin{bmatrix} 7 & 0 \\ 0 & -4 \end{bmatrix}$ $\quad P = \begin{bmatrix} 1 & -6 \\ 1 & 5 \end{bmatrix}$

$$A^2 = PDP^{-1}PDP^{-1}$$
$$= PDDP^{-1}$$
$$= PD^2P^{-1} \rightarrow \begin{bmatrix} 7 & 0 \\ 0 & -4 \end{bmatrix}\begin{bmatrix} 7 & 0 \\ 0 & -4 \end{bmatrix} = \begin{bmatrix} 49 & 0 \\ 0 & 16 \end{bmatrix}$$

$$A^3 = PDP^{-1}PDP^{-1}PDP^{-1}$$
$$= PD^3P^{-1}$$

$A^n = PD^nP^{-1}$ $\quad D^n = \begin{bmatrix} 7^n & 0 \\ 0 & (-4)^n \end{bmatrix}$

$$A^n = \begin{bmatrix} 1 & -6 \\ 1 & 5 \end{bmatrix}\begin{bmatrix} 7^n & 0 \\ 0 & (-4)^n \end{bmatrix}\begin{bmatrix} 1 & -6 \\ 1 & 5 \end{bmatrix}^{-1}$$

Diagonalization for general A (n×n)

1st Find characteristic eqn

$(A - \lambda I)$

2nd Find roots to $\det(A - \lambda I) = 0$

3rd Find eigen vectors

4th Write $A = PDP^{-1}$

What if we don't have n distinct roots of $\det(A - \lambda I) = 0$?

e.g. $(\lambda - 1)^2(\lambda - 2) = 0$

1 is multiple root

Thm A is diagonalizable iff

① all roots of $\det(A - \lambda I) \in \mathbb{R}$ → only important b/c we work over reals

② for each root λ_i, we want $\dim(\text{eigenspace}(\lambda_i))$ = multiplicity of lambda