§5.1 Eigenvectors & Eigenvalues
§5.2 Characteristic Polynomials

e.g. $A = \begin{bmatrix} 1 & 6 \\ 5 & 2 \end{bmatrix}$, $\vec{u} = \begin{bmatrix} 6 \\ -5 \end{bmatrix}$

$$A\vec{u} = \begin{bmatrix} 6-30 \\ 30-10 \end{bmatrix} = \begin{bmatrix} -24 \\ 20 \end{bmatrix} = -4\begin{bmatrix} 6 \\ -5 \end{bmatrix}$$

multiple of u.

$A\vec{u} = -4\vec{u}$.

Dfn A (n×n) matrix
(1) An eigenvector of A is a nonzero vector $\vec{x} \in \mathbb{R}^n$ such that $A\vec{x} = \lambda\vec{x}$ for some $\lambda \in \mathbb{R}$.
(2) In the above situation, we call λ the eigenvalue.
(3) If λ is an eigenvalue of A, then the solution set of $A\vec{x} = \lambda\vec{x}$ is called the eigenspace associated to λ.

e.g. $A = \begin{bmatrix} 2 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 3 \end{bmatrix}$ eigenvalues: $\lambda = 2$, $\lambda = 3$ (diagonal entries)

Computing the eigenspaces:
$\lambda = 2$ – solve $A\vec{x} = 2\vec{x}$

$$\begin{bmatrix} 2 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 3 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = 2\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} \quad \begin{cases} 2x_1 = 2x_1 \\ 2x_2 = 2x_2 \\ 3x_3 = 2x_3 \end{cases}$$

$x_3 = 0$

$$\begin{cases} 2x_1 = 2x_1 \\ 2x_2 = 2x_2 \\ 3x_3 = 2x_3 \end{cases}$$

solution set:

$$\text{span}\left(\left\{\begin{bmatrix}1\\0\\0\end{bmatrix}, \begin{bmatrix}1\\1\\0\end{bmatrix}\right\}\right)$$

This is the eigenspace associated to the eigenvalue 2. (dim = 2)

$\underline{\lambda = 3}$: $A\vec{x} = 3\vec{x}$

$$\begin{cases} 2x_1 = 3x_1 \\ 2x_2 = 3x_2 \\ 3x_3 = 3x_3 \end{cases}$$

Sol set: $\text{span}\left(\left\{\begin{bmatrix}0\\0\\1\end{bmatrix}\right\}\right)$ (dim = 1)

<u>Thm</u> If $\vec{x}_1, \dots, \vec{x}_r$ are eigenvectors of A with <u>distinct eigenvalues</u>, then $\vec{x}_1, \dots, \vec{x}_r$ are <u>linearly independent</u>.

ex: $A\vec{x}_1 = \lambda_1\vec{x}_1$, $A\vec{x}_2 = \lambda_2\vec{x}_2$ $(\vec{x}_1, \vec{x}_2 \neq \vec{0})$

$\lambda_1 \neq \lambda_2$

Then, $\vec{x}_1, \vec{x}_2$ are linearly independent.

Need to prove: (E1)

If $a_1\vec{x}_1 + a_2\vec{x}_2 = \vec{0}$, then $a_1 = a_2 = 0$. (linear independ)

Left Multiply A to both sides:

$A(a_1\vec{x}_1 + a_2\vec{x}_2) = \vec{0}$

$a_1 A\vec{x}_1 + a_2 A\vec{x}_2 = \vec{0}$

$a_1\lambda_1\vec{x}_1 + a_2\lambda_2\vec{x}_2 = \vec{0}$. (E2)

$\rightarrow E_2 - \lambda_1 E_1$

$= -a_2\lambda_1\vec{x}_2 + a_2\lambda_2\vec{x}_2 = \vec{0}$

$= a_2(\lambda_2 - \lambda_1)\vec{x}_2 = \vec{0}$

$\lambda_2 - \lambda_1 \neq 0$ (eigenval)

$\vec{x}_2 \neq 0$ (eigenvec $\neq \vec{0}$)

$\therefore a_2 = 0$.

$\rightarrow E_2 - \lambda_2 E_1$

$a_1 \lambda_1 \vec{x}_1 - a_1 \lambda_2 \vec{x}_1 = \vec{0}$

$a_1(\lambda_1 - \lambda_2)\vec{x}_1 = \vec{0}$ $\rightarrow$ $\lambda_1 - \lambda_2 \neq 0$

$\vec{x}_1 \neq 0$

$\therefore a_1 = 0.$

How to find eigens?

Start with $A\vec{x} = \lambda\vec{x}$

$\begin{pmatrix} A \ n \times n \\ I_n \ n \times n \end{pmatrix}$ $A\vec{x} = \lambda I_n \vec{x}$

$A\vec{x} - \lambda I_n \vec{x} = \vec{0}$

$(A - \lambda I_n)\vec{x} = \vec{0}$ Has nonzero solution $\vec{x}$.

so $\det(A - \lambda I_n) = 0$

$(\dim(\text{nul}(A - \lambda I_n)) \geq 1$

$\Rightarrow \text{rank}(A - \lambda I_n) < n$

$\Rightarrow \underbrace{\det(A - \lambda I_n) = 0}_{\text{characteristic equation}})$

Any eigenvalue is a root of the characteristic equation.

e.g. A $n \times n$ diagonal

$$A = \begin{bmatrix} a_1 & & 0 \\ & a_2 & \\ & & \ddots & \\ 0 & & & a_n \end{bmatrix} \quad A - \lambda I_n = \begin{bmatrix} a_1 - \lambda & & 0 \\ & a_2 - \lambda & \\ & & \ddots & \\ 0 & & & a_n - \lambda \end{bmatrix}$$

So Roots are $\lambda = a_1, \lambda = a_2, \ldots, \lambda = a_n$

Conclusion: diagonal matrix eigenvalues = diagonal matrix entries.