§4.5 Dimension
§4.6 Rank

Dfn V vector space, $\dim V$ (dimension of V) is # of elements in a basis of V.

ex $V = \mathbb{R}^n$, $\mathcal{B} = \{\vec{e}_1, \dots, \vec{e}_n\}$, $\dim V = n$.

ex $\mathbb{P}_n = \{\text{polynomials of deg} \le n\}$
$= a_0 + a_1 t + \dots + a_n t^n$
$\mathcal{B} = \{1, t, t^2, \dots, t^n\}$
$\dim \mathbb{P}_n = n+1$

ex $\mathbb{P} = \{\text{all polynomials}\}$ $\mathcal{B} = \{1, t, t^2, \dots\}$
$\dim \mathbb{P} = \infty$!

ex V vector space, $\mathcal{B} = \{\vec{0}\}$, $\dim V = 0$.

Thm Any basis of V has the same # of elements - $\dim V$ doesn't depend on basis chosen

Thm V vector space, $H \subset V$, then any basis of H can be extendended to a basis of V.
↳ by putting more vectors in basis

Consequently, $\dim H \le \dim V$.
$\dim H = \dim V \Leftrightarrow H = V$.

Thm V vector space, $\dim V = n$. then,
① Any n lin independent vectors form a basis for V.
② Any n vectors spanning V also form a basis.

A $(m \times n)$ matrix, $A = (\vec{a}_1, \dots, \vec{a}_n)$
$\text{nul}(A)$ = sol set of $A\vec{x} = \vec{0} \subseteq \mathbb{R}^n$.

$\text{col}(A) = \text{span}\{\vec{a}_1, \dots, \vec{a}_n\} \subseteq \mathbb{R}^m$.

Dfn $\text{rank}(A) = \dim \text{col}(A)$.

How to compute $\text{rank}(A)$ & $\dim \text{null}(A)$?

ex.

$$A = \begin{bmatrix} 1 & 6 & 2 & -4 \\ -3 & 2 & -2 & -8 \\ 4 & -1 & 3 & 9 \end{bmatrix}$$

$(m \times n) \to (3 \times 4)$

- row op to ref.
- get pivots
- # pivots = rank.
- $n - \text{rank}(A) = \dim \text{null}(A)$
  (# free variables)

General $m \times n$ A

$\dim \text{col}(A) = (\#\text{ of pivots})$
$\dim \text{nul}(A) = (\#\text{ free var})$
$\to \dim \text{col}(A) + \text{nul}(A) = n$.

Also ~~dim~~ $\text{rank}(A) = \dim \text{rowspace}(A)$
and $\text{rank}(A) = \text{rank}(A^T)$ for square matrix