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Recall A basis of a vector space V is a set $\{v_1, \dots, v_p\} \subset V$.

① it is linearly independent

② spans V.

ex. Find bases for null(A) & col(A).

$$A = \begin{bmatrix} 1 & 4 & 0 & 2 \\ 3 & 12 & 1 & 5 \\ 2 & 8 & 1 & 3 \\ 5 & 20 & 2 & 8 \end{bmatrix}$$

Sol

First, find echelon form.

$$\Rightarrow B = \begin{bmatrix} \boxed{1} & 4 & 0 & 2 \\ 0 & 0 & \boxed{1} & -1 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}.$$

Then, find sol. set:

$x_1 = -4x_2 \quad -2x_4$

$x_2 = x_2$

$x_3 = \quad x_4$

$x_4 = \quad x_4$

sol set:

$$\text{span}\left(\left\{ \underbrace{\begin{bmatrix} -4 \\ 1 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} -2 \\ 0 \\ 1 \\ 1 \end{bmatrix}}_{\text{basis}} \right\}\right) \Leftarrow$$

columnspace: → basis.

$$\text{span}\left(\left\{ \begin{bmatrix} 1 \\ 3 \\ 2 \\ 5 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \\ 1 \\ 2 \end{bmatrix} \right\}\right)$$

original columns of pivots

Reason: row operations do not change column relations

Two different characteristics of a basis of V:

(i) smallest set that spans V

(ii) largest set that is linearly independent.

§4.4 Coordinate Systems

V vectorspace

$\mathcal{B} = \{\vec{v}_1, \dots, \vec{v}_p\}$ basis of V

Then, any vector $\vec{u} \in V$ is of the form:

$$\vec{u} = c_1\vec{v}_1 + c_2\vec{v}_2 + \dots + c_p\vec{v}_p$$

$\begin{bmatrix} c_1 \\ \vdots \\ c_p \end{bmatrix}$ are the coordinates of $\vec{u}$, with respect to the basis $\mathcal{B}$. (convention: coordinates are written as a column vector)

ex. $\mathbb{R}^n$, standard basis:

$$\mathcal{B} = \left\{ \begin{bmatrix} 1 \\ 0 \\ \vdots \\ 0 \end{bmatrix}, \dots, \begin{bmatrix} 0 \\ 0 \\ \vdots \\ 1 \end{bmatrix} \right\}$$

coordinates of

if $\vec{b} = \begin{bmatrix} b_1 \\ \vdots \\ b_n \end{bmatrix} \in \mathbb{R}^n$, then $[\vec{b}]_{\mathcal{B}} = \begin{bmatrix} b_1 \\ \vdots \\ b_n \end{bmatrix}$

<u>ex</u> $\vec{b}_1 = \begin{bmatrix} 1 \\ 2 \end{bmatrix}$ $\vec{b}_2 = \begin{bmatrix} 0 \\ 1 \end{bmatrix}$ $\Rightarrow \mathcal{B}$

$\vec{x} = \begin{bmatrix} 3 \\ 4 \end{bmatrix}$, find $[x]_{\mathcal{B}}$.

$$\begin{array}{ccc} b_1 & b_2 & \vec{x} \\ \end{array}$$
$$\underbrace{\begin{bmatrix} 1 & 0 \\ 2 & 1 \end{bmatrix}}_{P_B}\begin{matrix} \Big| \end{matrix}\begin{bmatrix} 3 \\ 4 \end{bmatrix} \Rightarrow \left[\begin{array}{cc|c} 1 & 0 & 3 \\ 0 & 1 & -2 \end{array}\right]$$

solution!

in other words, $[\vec{x}]_{\mathcal{B}} = P_{\mathcal{B}}^{-1} x$

Coordinate Mapping: $\mathcal{B} = \{\vec{v}_1, \dots, \vec{v}_p\}$ basis of V

$T: V \to \mathbb{R}^p$ — linear transformation that is bijective called an isomorphism.

$T(\vec{u}) = [\vec{u}]_{\mathcal{B}}$

(vector to its coordinate)

Def V is a vector space The dimension of V, dim(v), is the # of elements in the basis of V.

Rmk dim(V) doesn't depend on basis chosen. All bases have same # of elements.