*Quiz next Tuesday (§4.1-4.6 in Discussion)

§4.3 Bases

V vector space.

$B = \{\vec{v}_1, \vec{v}_2, \ldots, \vec{v}_p\}$ set of vectors in V

Say $\vec{v}_1, \ldots, \vec{v}_p$ is linearly independent if only possible solution to $x_1\vec{v}_1 + \ldots + x_p\vec{v}_p$ is $x_1, \ldots, x_p = 0$. (no redundant relations)

Say $\{\vec{v}_1, \ldots, \vec{v}_p\}$ is a basis of V if

① they are linearly independent

② they span V. (any vector in V is a linear combo of $\{\vec{v}_1, \ldots, \vec{v}_p\}$)

ex. standard basis of $\mathbb{R}^n$

$$\vec{e}_1 = \begin{bmatrix} 1 \\ 0 \\ \vdots \\ 0 \end{bmatrix}, \quad \vec{e}_2 = \begin{bmatrix} 0 \\ 1 \\ \vdots \\ 0 \end{bmatrix} \ldots \quad e_n = \begin{bmatrix} 0 \\ 0 \\ \vdots \\ 1 \end{bmatrix}$$

$\mathcal{B} = \{\vec{e}_1, \ldots, \vec{e}_n\}$

① these vectors are linearly dependent.

② they span $\mathbb{R}^n \rightarrow$ e.g. $\begin{bmatrix} a_1 \\ \vdots \\ a_n \end{bmatrix} = a_1 \begin{bmatrix} 1 \\ 0 \\ \vdots \\ 0 \end{bmatrix} + \ldots + a_n \begin{bmatrix} 0 \\ \vdots \\ 1 \end{bmatrix}$

ex. $P_n = \{$polynomials w/ degree $\leq n\}$ has the standard basis $\{1, t, t^2, \ldots, t^n\}$:)

Thm $\{\vec{v}_1, \ldots, \vec{v}_n\}$ vectors of $\mathbb{R}^n$. Then the set is a basis of $\mathbb{R}^n$ iff the matrix $A = [\vec{v}_1, \vec{v}_2 \ \ldots \ \vec{v}_n]$ = identity matrix.

Prf Check if basis is invertible

① lin indep. the equation $x_1\vec{v}_1 + \ldots + x_n\vec{v}_n = 0$ is

$$[\vec{v}_1 \ \ldots \ \vec{v}_n]\begin{bmatrix} x_1 \\ \vdots \\ x_n \end{bmatrix} = 0$$

if A invertible, only 0 is solution.

② span: $b \in V$ any vector

want solution where: $x_1\vec{v}_1 + \ldots + x_n\vec{v}_n = \vec{b}$.

(This is just $A\vec{x} = \vec{b}$ – * has unique solution $\forall \vec{b}$)

Thm Spanning Set:

V vector space $\vec{v}_1, \ldots, \vec{v}_s \in V$

$H = \text{span}(\{\vec{v}_1, \ldots, \vec{v}_s\})$ subspace.

Then, there is a subset of $\{\vec{v}_1, \ldots, \vec{v}_p\}$ which is a basis of H.

Algorithm:

if $\vec{v}_1$ is nonzero, add to S. (else delete it)

if S & $\vec{v}_2$ are linearly dep., add $\vec{v}_2$ to S (else delete it)

if S & $\vec{v}_3$ " " $\vec{v}_3$ "

⋮

up to $\vec{v}_s$. You get a basis. Hurray!

Basis for Nullspace & Columnspace

A $(m \times n)$

nullspace(A) $= \{\vec{x} \in \mathbb{R}^n \mid A\vec{x} = \vec{0}\}$.

$\Rightarrow$ sol set is lin. combo of vectors which is the basis.

columnspace(A) $=$ span($\{$columns of A$\}$)

$\Rightarrow$ basis is set of all linearly independent column vectors. (pivot columns of A, NOT the reduced echelon form)