§4.2 Column Space & Null Space

Recall A vector space is a set of vectors that supports addition and scalining and satisfies: ① Has $\vec{0}$ ② commutative $(+\cdot)$ ③ associative $(+\cdot)$ ④ distributive $(+\cdot)$

Key $\vec{u}, \vec{v} \in V \Rightarrow \vec{u} + \vec{v} \in V$.

$a\vec{u} \in V \ \forall a \in \mathbb{R}$

V is a vectorspace.

$\vec{v}_1, \ldots, \vec{v}_p \in V$.

A linear combination of $\vec{v}_1, \ldots, \vec{v}_p$ is a vector of the form $a_1\vec{v}_1 + \ldots + a_p\vec{v}_p$ ($a_1, \ldots, a_p$ scalars)

The subspace spanned by $\vec{v}_1, \ldots, \vec{v}_p$ is the set of all linear combinations of $\vec{v}_1, \ldots, \vec{v}_p$ (by varying the scalars).

This is denoted by: $\text{span}\{\vec{v}_1, \ldots, \vec{v}_p\}$

called spanning set or generator.

Def A $m \times n$ matrix

$$
\begin{aligned}
\text{col}(A) &= \text{column space of } A \\
&= \text{subspace of } \mathbb{R}^m \text{ spanned by columns of } A \\
&= \text{span}(\{\vec{a}_1, \ldots, \vec{a}_n\}) \subset \mathbb{R}^m \\
&\qquad (\text{if } A = [\vec{a}_1 \ldots \vec{a}_n])
\end{aligned}
$$

$$
\begin{aligned}
\text{null}(A) &= \text{null space of } A \\
&= \text{solution set of system } A\vec{x} = \vec{0} \\
&= \{\vec{x} \in \mathbb{R}^n \mid A\vec{x} = \vec{0}\} \\
&= \left\{ \begin{bmatrix} x_1 \\ \vdots \\ x_n \end{bmatrix} \in \mathbb{R}^n \;\middle|\; x_1\vec{a}_1 + \ldots x_n\vec{a}_n = 0 \right\} \subset \mathbb{R}^n
\end{aligned}
$$

e.x $A = \begin{bmatrix} 1 & 0 & 5 & 6 \\ 0 & 1 & 2 & 3 \end{bmatrix}$

① null(A) = ?

$x_1 = -5x_3 - 6x_4$
$x_2 = -2x_3 - 3x_4$
$\Rightarrow \text{span}\left(\left\{ \begin{bmatrix} -5 \\ -2 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} -6 \\ -3 \\ 0 \\ 1 \end{bmatrix} \right\}\right)$

② col(A) = ?

$\begin{bmatrix} 5 \\ 2 \end{bmatrix}$ and $\begin{bmatrix} 6 \\ 3 \end{bmatrix}$ can be expressed by linear combo of $\begin{bmatrix} 1 \\ 0 \end{bmatrix}$ and $\begin{bmatrix} 0 \\ 1 \end{bmatrix}$.

$\therefore \text{col}(A) = \text{span}\left(\left\{ \begin{bmatrix} 0 \\ 1 \end{bmatrix}, \begin{bmatrix} 1 \\ 0 \end{bmatrix} \right\}\right) \subseteq \mathbb{R}^2$

Linear Transformation

$T(\vec{x}) = A\vec{x} \quad (T: \mathbb{R}^n \to \mathbb{R}^m)$

Kernal (nullspace) $(T) = \{\vec{x} \in \mathbb{R}^n \mid T(\vec{x}) = 0\}$.

= "pre-image" of 0

Range(T) = {all values of T} = columnspace

Def (more general lin. trans)

V, W vector spaces

A linear transformation is a map $T: V \to W$ satisfying

① $T(\vec{u} + \vec{v}) = T(\vec{u}) + T(\vec{v})$

② $T(a\vec{v}) = aT(\vec{v})$.

ex. V = {all continuous functions on $\mathbb{R}$}

$W = \mathbb{R}$

$T: V \to W$

$f \mapsto f(2)$