§ 4.1 Vector Spaces (conceptual)

Key: Definitions (Dfn)

Dfn: A vector space a set of vectors with two operations called addition and scalar multiplication.

addition: $\vec{u}, \vec{v} \in V$, then $\vec{u} + \vec{v} \in V$

multiplication: $\vec{u} \in V$, then $\alpha \vec{u} \in V \; \forall \alpha \in \mathbb{R}$.

Vector spaces satisfy the following properties:

1. $\vec{u} + \vec{v} = \vec{v} + \vec{u}$ (commutative)
2. $(\vec{u} + \vec{v}) + \vec{w} = \vec{u} + (\vec{v} + \vec{w})$ (associative)
3. $\vec{0}$ in V. $\vec{0} + \vec{v} = \vec{v} = \vec{v} + \vec{0}$.
4. $a(\vec{u} + \vec{v}) = a\vec{u} + a\vec{v}$
5. $(a+b)\vec{u} = a\vec{u} + b\vec{u}$

$\Big\}$ distributive

6. $(ab)\vec{u} = a(b\vec{u})$
7. $1\vec{u} = \vec{u}$

ex. $\mathbb{R}^n$ is a vector space.

ex. Solution of the eqn $A\vec{x} = \vec{0}$ (where A is $m \times n$ matrix, $\vec{x}$ vector in $\mathbb{R}^n$) is a vector space

Reason: If x_1, x_2 are solutions, then $x_1 + x_2$ is a solution. Similar for scalar mult.

Rmk: subtraction? It is derived from addition and scalar multiplication. $\vec{u} - \vec{v} = \vec{u} + (-1)\vec{v}$

ex. Solution set of $x_1 + x_2 = 1$ is not a vector space.

1. no $\vec{0}$.
2. does not scale or add ($\vec{u} + \vec{v} \notin V$)

Dfn If V is a vector space, a <u>subspace</u> of V is a subset H of V satisfying

① $\vec{0} \in H$

② H closed under addition ($\vec{u}, \vec{v} \in H \Rightarrow \vec{u}+\vec{v} \in H$)

③ H closed under scalar multiplication ($\vec{u} \in H$, $\alpha\vec{u} \in H$, $\alpha \in \mathbb{R}$)

Thm. In the definition above, H is a vector space.

ex. A ($m \times n$) matrix, $V = \mathbb{R}^n$

H solution set $A\vec{x} = \vec{0}$. $H = \{\vec{x} \in \mathbb{R}^n : A\vec{x} = \vec{0}\}$

$H \subset \mathbb{R}^n$, and is a subspace of $\mathbb{R}^n$.

ex $V = \{\text{Continuous functions } \mathbb{R} \to \mathbb{R}\}$

is a vector space.

check: f, g cont $\Rightarrow$ f+g cont.

f cont, $a \in \mathbb{R} \Rightarrow af$ cont.

$\vec{0}$ constant function, 0 in V.

Thm V vector space. $\vec{v}_1, \ldots, \vec{v}_p \in V$

Then span$\{\vec{v}_1, \ldots, \vec{v}_p\}$ is a subspace of V.

Here, span $\{\vec{v}_1, \ldots, \vec{v}_p\}$

$= \{a_1\vec{v}_1 + \cdots + a_p v_p \mid a_1, \ldots, a_p \in \mathbb{R}\}$

= all vectors of form $a_1\vec{v}_1 + \ldots + a_p\vec{v}_p$.