$$x + 2y + az = 0$$
$$-x \quad + z = 0$$
$$ax - y + z = 0$$

$$\begin{bmatrix} 1 & 2 & a \\ -1 & 0 & 1 \\ a & -1 & 1 \end{bmatrix}$$

2(

$$\sim \begin{bmatrix} 1 & 0 & -1 \\ 0 & 2 & a+1 \\ 0 & -1-2a & 1-a^2 \end{bmatrix}$$

$$\sim \begin{bmatrix} 1 & 2 & a \\ 0 & 2 & 1+a \\ 0 & 0 & 2+\frac{a}{2} \end{bmatrix}$$

one sol:

inf. sol: $a = -4$

<u>Dfn</u> $\{\vec{v}_1, \ldots, \vec{v}_n\} \subseteq V$

$\text{span}(\{\vec{v}_1, \ldots, \vec{v}_n\})$ = all linear combos of those vectors

<u>Prp</u> $S \subseteq V$ a subset of v.space. Then $\exists!$ subspace of V, $\text{span}(S)$, s.t.

(i) $S \subseteq \text{span}(S)$

(ii) $W \subseteq V$ subspace w/ $S \subseteq W$, then $\text{span}(S) \subseteq W$.

$\text{span}(\{\vec{v}_1, \vec{v}_2, \vec{v}_3, \vec{v}_4\}) = \text{span}(\{\vec{v}_1, \vec{v}_2, \vec{v}_3\})$

$\Downarrow$

<u>Prp</u> $\vec{v}_4 \in \text{span}(\{\vec{v}_1, \vec{v}_2, \vec{v}_3\})$

Prf: Proceed by contradiction: assume $\vec{v}_4 \notin \text{span}(\{\vec{v}_1, \vec{v}_2, \vec{v}_3\})$

Then, $\text{Span}(\{\vec{v}_1, \ldots, \vec{v}_4\}) \supset \text{Span}(\{\vec{v}_1, \ldots, \vec{v}_3\})$

because $\vec{v}_4 \in \text{span}(\{\vec{v}_1, \ldots, \vec{v}_3\}) \leftarrow$ contradiction.

~~Given 3 vectors, prove~~

$\mathcal{B} = \{1, T-1, (T-1)^2\} \subseteq \mathbb{R}[x]/(x^3)$

$\mathcal{B} = \{1, x, x^2\}$

$[1]_{\mathcal{B}} = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} \quad [t-1]_{\mathcal{B}} = \begin{bmatrix} -1 \\ 1 \\ 0 \end{bmatrix} \quad [(t-1)^2]_{\mathcal{B}} = \begin{bmatrix} 1 \\ -2 \\ 1 \end{bmatrix}$

row reduce! $\begin{bmatrix} 1 & 0 & 0 \\ -1 & 1 & 0 \\ 1 & -2 & 1 \end{bmatrix} \sim \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & -2 & 1 \end{bmatrix}$

$\sim \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$ sufficient basis-ness.

$\begin{bmatrix} 1 & 0 & 0 \\ 1 & 1 & 1 \\ 0 & 1 & 1 \end{bmatrix}$

Linear independence

$\{v_1, \ldots, v_n\} \subseteq V$ lin. ind. iff

$\alpha_1 v_1 + \ldots + \alpha_n v_m = 0 \Rightarrow \alpha_1 = 0, \ldots, \alpha_m = 0$

Basis

$\mathcal{B} := \{\vec{v}_1, \ldots, \vec{v}_m\}$ iff (i) $\mathcal{B}$ is lin. ind.

(ii) $\operatorname{span}(\mathcal{B}) = V$

Dim:

$\mathcal{B}_1, \mathcal{B}_2$ bases $\Rightarrow |\mathcal{B}_1| = |\mathcal{B}_2| =: \dim(V)$

Rank Nullity:

$T: V \to W$ lin. trans, $\dim(V), \dim(W) < \infty$

$\dim(\operatorname{Img}(T)) + \dim(\operatorname{Ker}(T)) = \dim(V)$

Prp $T: V \to W$ lin trans. $\mathcal{B} = \{b_1, \dots, b_d\}$ is V basis
$\mathcal{C} = \{c_1, \dots, c_e\}$ is W basis

Then, $\exists!$ $e \times d$ matrix $[T]_{\mathcal{C} \leftarrow \mathcal{B}}$ s.t.
$[T(v)]_{\mathcal{C}} = [T]_{\mathcal{C} \leftarrow \mathcal{B}} [v]_{\mathcal{B}} \ \forall v \in V$

Furthermore,
$[T]_{\mathcal{C} \leftarrow \mathcal{B}} = [[T(b_1)]_{\mathcal{C}} \cdots [T(b_d)]_{\mathcal{C}}]$

$T: \mathbb{R}^3 \to \mathbb{R}^3$, Find $[T]$.

$$T\left(\begin{bmatrix}1\\1\\1\end{bmatrix}\right) = \vec{e}_1 = T(e_1) + T(e_2) + T(e_3) \qquad [T] = [T(\vec{e}_1), T(\vec{e}_2), T(\vec{e}_3)]$$

$$T\left(\begin{bmatrix}1\\0\\1\end{bmatrix}\right) = \vec{e}_2 = T(e_1) + T(e_3)$$

$$T\left(\begin{bmatrix}0\\-1\\1\end{bmatrix}\right) = \vec{e}_3 \quad -T(e_2) + T(e_3)$$

$$\begin{bmatrix}e_1\\e_2\\e_3\end{bmatrix} = \begin{bmatrix}1&1&1\\1&0&1\\0&-1&1\end{bmatrix}\begin{bmatrix}T(e_1)\\T(e_2)\\T(e_3)\end{bmatrix}$$

$$\left[\begin{array}{ccc|c}1&1&1&e_1\\1&0&1&e_2\\0&-1&1&e_3\end{array}\right] \sim \left[\begin{array}{ccc|c}1&1&1&e_1\\0&-1&0&-e_1+e_2\\0&-1&1&e_3\end{array}\right]$$

$$\sim \left[\begin{array}{ccc|c}1&1&1&e_1\\0&+1&0&e_1-e_2\\0&0&1&e_3+e_1-e_2\end{array}\right] \sim \left[\begin{array}{ccc|c}1&0&0&2e_2-e_3-e_1\\0&1&0&e_1-e_2\\0&0&1&e_3+e_1-e_2\end{array}\right]$$

$$T(\vec{e}_1) = \begin{bmatrix}-1\\2\\-1\end{bmatrix} \quad T(\vec{e}_2) = \begin{bmatrix}1\\-1\\0\end{bmatrix} \quad T(\vec{e}_3) = \begin{bmatrix}1\\-1\\1\end{bmatrix}$$

Prp Let $f : X \to Y$ be a func between nonempty sets
Then, -
(i) f is inj. iff f has left inverse
(ii) f is surj iff f has right inverse
(i.e. $\exists\, g,\ Y \to X$ s.t.
(i) $g \circ f = id_X$
(ii) $f \circ g = id_Y$

$BAB = A$
$B(ABA^{-1}) = id$
$\Rightarrow T_B$ is surj.
$\Rightarrow T_B$ is bijective (ranknullity - it B square) $\Rightarrow T_B$ is invertible

$T : \mathbb{R}^3 \to \mathbb{R}^2$, $S : \mathbb{R}^2 \to \mathbb{R}^3$
$S \circ T : \mathbb{R}^3 \to \mathbb{R}^3$ is not invertible.
$Im(S \circ T) \subseteq Im(S)$
$\dim(Im(S \circ T)) \leq \dim(Im(S)) \leq 2 < 3$.

$T \circ S : \mathbb{R}^2 \to \mathbb{R}^2$ can be invertible.

$T = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \end{bmatrix}$ $S = \begin{bmatrix} 1 & 0 \\ 0 & 1 \\ 0 & 0 \end{bmatrix}$

$TS = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$

Dfn det: $M_{nn}(\mathbb{R}) \to \mathbb{R}$ is unique fn s.t.

(i) $\det(id) = 1$

(ii) $\det(E_{i \leftrightarrow j} A) = -\det(A)$

(iii) $\det(E_{r \to ar} A) = a \cdot \det(A)$, $a \in \mathbb{R}$, $a \neq 0$

(iv) $\det(E_{r_i \to r_i + ar_j} A) = \det(A)$

$adj(A) := Cof(A)^T$

$A\, adj(A) = \det(A)\, id = A\, adj(A)$

$T: \mathbb{R}^2 \to T: \mathbb{R}^2$

rotate $\pi/4$ rad cc aboutorigin

reflect acceross y-axis

standard matrix $\to$?

$\frac{\sqrt{2}}{2}$,

$[T] = [T(e_1) \quad T(e_2)]$

$T\begin{pmatrix} 1 \\ 0 \end{pmatrix} = \begin{pmatrix} -\sqrt{2}/2 \\ +\sqrt{2}/2 \end{pmatrix}$

$T\begin{pmatrix} 0 \\ 1 \end{pmatrix} = \begin{pmatrix} +\sqrt{2}/2 \\ \sqrt{2}/2 \end{pmatrix}$

$$= \begin{bmatrix} -\sqrt{2}/2 & \sqrt{2}/2 \\ \sqrt{2}/2 & \sqrt{2}/2 \end{bmatrix}$$