§3.2 Properties of Determinants

02/10

Upper triangular matrix:

(square matrix, every entry below diagonal is 0)

$$\det\left(\begin{bmatrix} a_{11} & & & \\ 0 & a_{22} & & \\ 0 & 0 & a_{33} & \cdots \\ 0 & 0 & & \end{bmatrix}\right) \leftarrow \text{echelon form}$$

$= a_{11} a_{22} \cdots a_{mm}$ (product of entries on diagonal)

e.g.

$$\det\begin{pmatrix} a & * & * & * \\ 0 & b & * & * \\ 0 & 0 & c & * \\ 0 & 0 & 0 & d \end{pmatrix} = abcd$$

$$= a \cdot \det\begin{pmatrix} b & * & * \\ 0 & c & * \\ 0 & 0 & d \end{pmatrix}$$

$$= a \cdot b \det\begin{pmatrix} c & * \\ 0 & d \end{pmatrix}$$

$$= abcd.$$

Row operations on determinant

① interchange two rows (or two columns), det(A) becomes -det(A).

② scaling 1 row by a constant, det(A) becomes c·det(A).

③ Add multiple of 1 row to another - det(A) does not change.

Same rules for column operations.

eg $\begin{bmatrix} 1 & -4 & 1 \\ 2 & -4 & 4 \\ 3 & -4 & 9 \end{bmatrix}$

$= -4 \begin{bmatrix} 1 & 1 & 1 \\ 2 & 1 & 4 \\ 3 & 1 & 9 \end{bmatrix}$ $\qquad \frac{\det}{-4}$

$= 4 \begin{bmatrix} 1 & 1 & 1 \\ 1 & 2 & 4 \\ 1 & 3 & 9 \end{bmatrix}$ $\qquad \frac{\det}{4}$

$= 4 \begin{bmatrix} 1 & 1 & 1 \\ 0 & 1 & 3 \\ 0 & 2 & 8 \end{bmatrix}$ $\qquad \frac{\det}{4}$

$\Updownarrow$

$= 4 \begin{bmatrix} 1 & 3 \\ 2 & 8 \end{bmatrix}$

$= 4(2) = \boxed{8}$ ✓

<u>Thm</u>: $\det(AB) = \det(A) \cdot \det(B)$

<u>Thm</u>: $\det(A^T) = \det(A)$

<u>Thm</u>. If A is invertible iff $\det(A) \neq 0$

Then, $\det(A^{-1}) = \det(A)^{-1}$.

§3.3 Cramer's Rule

Thm: A $(n\times n)$ invertible matrix. Then, the equation

$A\vec{x}=\vec{b}$ has a unique solution

Remember. $A\vec{x}=\vec{b}$

$$A^{-1}\cdot A\vec{x}=A^{-1}\vec{b}$$

$$\vec{x}=A^{-1}\vec{b}$$

but the unique sol can also be calculated as:

$$\begin{cases} x_1=\dfrac{\det(A_1(\vec{b}))}{\det(A)} \\ \vdots \\ x_n=\dfrac{\det(A_n(\vec{b}))}{\det(A)} \end{cases}$$

where for x_i, the matrix $A_i(\vec{b})$ is the matrix obtained by replacing the ith column of A with $\vec{b}$.

e.g. $\begin{cases} x_1-5x_2=3 \\ -2x_1+3x_2=2 \end{cases}$

$$\underbrace{\begin{bmatrix} 1 & -5 \\ -2 & 3 \end{bmatrix}}_{A} \; \underbrace{\begin{matrix} 3 \\ 2 \end{matrix}}_{\vec{b}}$$

reduce, etc. or ...

use Cramer's Rule:

$A_1(b)=\begin{bmatrix} 3 & -5 \\ 2 & 3 \end{bmatrix}$ $\det(A_1\vec{b})=19$

$A_2(b)=\begin{bmatrix} 1 & 3 \\ -2 & 2 \end{bmatrix}$ $\det(A_2(\vec{b}))=8$

$\det(A)=-7$

Solution:

$x_1=\dfrac{-19}{7}$.

$x_2=\dfrac{-8}{7}$.

Reason: Perform row operation on $[A \ \vec{b}]$

get $[I_n \ \vec{b}]$

Thm If A is invertible, then $A^{-1} = \frac{1}{\det(A)} \begin{pmatrix} C_{11} & \cdots & C_{1n} \\ C_{21} & & C_{2n} \\ \vdots & & \\ C_{n1} & \cdots & C_{nn} \end{pmatrix}$

where the (i,j) entry $C_{ij} = (-1)^{i+j} \det(A_{ij})$

A_{ij} is A deleting i-th row, j-th col.

Recall: C_{ij} the cofactor of A_{ij}

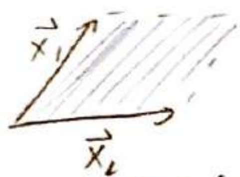
$\det(A) = a_{i1} C_{i1} + a_{i2} C_{i2} + \cdots + a_{in} C_{in}$

Proof of formula of A^{-1}

Check $A A^{-1} = I_n$

track every entry.

Areas in $\mathbb{R}^2$:



is given by $|\det(x_1 \ x_2)|$.