Linear Transformation:

$\alpha T(v) = T(\alpha v)$ & $T(x+y) = T(x) + T(y)$

Subspace:

i) $\vec{0} \in S$

ii) $v, w \Rightarrow v+w \in S$

iii) $v \in S \Rightarrow \alpha v \in S$

<u>Dfn</u> $T: V \to W$

$Ker(T) := \{v \in V : T(v) = \vec{0}\}$

$Im(T) := \{T(v) : v \in V\}$

<u>Prp</u> Kernal is subspace of domain

Image is subspace of codomain

$T: \mathbb{R}^5 \to \mathbb{R}^2$

$T(x) := Ax$, $A = \begin{bmatrix} 1 & 2 & 0 & -3 & 5 \\ 2 & 4 & 0 & 6 & 1 \end{bmatrix}$

$Ker(T) = ?$ Use basis language.

$Img(T) = ?$

$$\begin{bmatrix} 1 & 2 & 0 & -3 & 5 \\ 0 & 0 & 0 & 12 & -9 \end{bmatrix}$$

$$\sim \begin{bmatrix} 1 & 2 & 0 & 0 & 11/4 \\ 0 & 0 & 0 & 1 & -3/4 \end{bmatrix} \therefore \quad \begin{aligned} x_4 &= \tfrac{3}{4} x_5 \\ x_1 &= -2x_2 - 11/4\, x_5 \end{aligned}$$

Basis: $\left\{ \begin{bmatrix} -2 \\ 1 \\ 0 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 0 \\ 1 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} -11/4 \\ 0 \\ 0 \\ 3/4 \\ 1 \end{bmatrix} \right\}$ $Ker(T) = span\left(\left\{ \begin{bmatrix} -2 \\ 1 \\ 0 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 0 \\ 1 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} -11/4 \\ 0 \\ 0 \\ 3/4 \\ 1 \end{bmatrix} \right\} \right)$

$Img(T)$ $Span\left(\left\{ \begin{bmatrix} 1 \\ 2 \end{bmatrix}, \begin{bmatrix} 3 \\ -6 \end{bmatrix} \right\} \right)$

$T: \mathbb{R}[x] \to \mathbb{R}[x]$

$T := d/dx$

$\ker(T) = \text{Span}(\{[1]\})$

$\text{Im}(T) = \text{span}(\{[1], [x], [x^2], \ldots, [x^n]\}) = \mathbb{R}[x]$

Determinants:

<u>Thm</u> $\exists!$ function $\det: M_{n \times n}(\mathbb{R}) \to \mathbb{R}$

i) $\det(id_{n \times n}) = 1$

ii) row swap changes sign

iii) scaling row scales det

iv) det. is invariant under "other row op" (adding rows)

$$\det\left(\begin{bmatrix} 2 & 4 \\ 8 & -2 \end{bmatrix}\right) = 4 \det\left(\begin{bmatrix} 1 & 2 \\ 4 & -1 \end{bmatrix}\right)$$ *scale rows, not matrices

<u>Prp</u> T inv. iff det(T) inv ($\det(T) \neq 0$)

<u>Prp</u> $S \subseteq \mathbb{R}^d$, $T: \mathbb{R}^d \to \mathbb{R}^d$ (lin trans)

$\text{vol}(T(S)) = |\det(T)| \text{vol}(S)$

<u>Cofuctors</u>

$$\begin{array}{c c} & \begin{matrix} + & - & + \end{matrix} \\ \begin{matrix} + \\ - \\ + \end{matrix} & \begin{bmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \\ 7 & 8 & 9 \end{bmatrix} \end{array} = A$$

$\text{cof}(A_{1,1}) = 45 - 48 = -3$

$\text{cof}(A_{3,2}) = -\det\begin{pmatrix} 1 & 3 \\ 4 & 6 \end{pmatrix}$

$\det(A) = 7 \cdot \text{cof}(A_{3,1}) = 7 \cdot \det\begin{pmatrix} 2 & 3 \\ 5 & 6 \end{pmatrix}$

$+ 8 \cdot \text{cof}(A_{3,2}) + {}^{-}8 \cdot \det\begin{pmatrix} 1 & 3 \\ 4 & 6 \end{pmatrix}$

$9 \cdot \text{cof}(A_{3,3}) + 9 \cdot \det\begin{pmatrix} 1 & 2 \\ 4 & 5 \end{pmatrix}$

$\text{cof}(A) = \Bigg[$

$\mathrm{adj}(A) := \mathrm{cof}(A)^T$

$A\,\mathrm{adj}(A) = \det(A)\,\mathrm{id} = \mathrm{adj}(A)\,A$

If A invertible $\Rightarrow$ $A^{-1} = \det(A)^{-1}\,\mathrm{adj}(A)$

Cramer's Rule:

$A = [\vec{a}_1 \dots \vec{a}_n]$, $\vec{b} = \begin{bmatrix} b_1 \\ \vdots \\ b_n \end{bmatrix}$, $\vec{x} = \begin{bmatrix} x_1 \\ \vdots \\ x_n \end{bmatrix}$

If A is invertible and $A\vec{x} = \vec{b}$, $x_k = \dfrac{\det(a_1 \dots a_{k-1}, b, a_{k+1} \dots a_n)}{\det(\vec{a} \dots \vec{a}_n)}$

e.g. $A = \begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix}$, $b = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$

$$x_1 = \frac{\det\begin{pmatrix} 1 & 2 \\ 1 & 4 \end{pmatrix}}{\det(A)} = \frac{2}{-2} = -1$$

$$x_2 = \frac{\det\begin{pmatrix} 1 & 1 \\ 3 & 1 \end{pmatrix}}{\det(A)} = \frac{-2}{-2} = 1$$