02/08

§2.3 Characterizations of Invertible Matrices

Thm Let A (n×n) matrix. The following are equivalent.

a. A is an invertible matrix

b. A is row equivalent to the n×n identity matrix

c. A has n pivot positions

d. The eqn $A\vec{x} = \vec{0}$ has only the trivial solution

e. The columns of A form a linearly independent set

f. The linear transformation $\vec{x} \mapsto A\vec{x}$ is one-to-one

g. The equation $A\vec{x} = \vec{b}$ has at least 1 solution for each $\vec{b}$ in $\mathbb{R}^n$

h. The columns of A

i. The linear trans. $\vec{x} \mapsto A\vec{x}$ maps $\mathbb{R}^n$ to $\mathbb{R}^n$

j. There is an n×n C such that $CA = I$.

k. There is an n×n D such that $AD = I$.

l. A^T is an invertible matrix.

invertible matrix: nonsingular

noninvertible matrix: singular

e.g. Is $A = \begin{bmatrix} 1 & 0 & -2 \\ 3 & 1 & -2 \\ -5 & -1 & 9 \end{bmatrix}$ invertible?

$$\sim \begin{bmatrix} 1 & 0 & -2 \\ 0 & 1 & 4 \\ 0 & -1 & 1 \end{bmatrix} \sim \begin{bmatrix} 1 & 0 & -2 \\ 0 & 1 & 4 \\ 0 & 0 & 3 \end{bmatrix}$$

3 pivots ∴ invertible.

Invertible Linear Transformations:

A linear transformation $T: \mathbb{R}^n \to \mathbb{R}^n$ is invertible if $\exists$ a function $S: \mathbb{R}^n \to \mathbb{R}^n$ such that:

$$S(T(\vec{x})) = \vec{x} \quad \forall \vec{x} \in \mathbb{R}^n$$
$$T(S(\vec{x})) = \vec{x} \quad \forall \vec{x} \in \mathbb{R}^n$$

- S is the inverse of T (T^{-1}).

Thm let $T: \mathbb{R}^n \to \mathbb{R}^n$ be a linear transformation and let A be the standard matrix. T is invertible iff A is an invertible matrix.
($T^{-1} = A^{-1}\vec{x}$)

§3.1 Introduction to Determinants

3×3 case - Consider $A = [a_{ij}]$ with $a_{11} \neq 0$.

$$\begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{11}a_{21} & a_{11}a_{22} & a_{11}a_{23} \\ a_{11}a_{31} & a_{11}a_{32} & a_{11}a_{33} \end{bmatrix} \sim \begin{bmatrix} a_{11} & a_{12} & a_{13} \\ 0 & a_{11}a_{22}-a_{12}a_{21} & a_{11}a_{23}-a_{13}a_{21} \\ 0 & a_{11}a_{32}-a_{12}a_{31} & a_{11}a_{33}-a_{13}a_{31} \end{bmatrix}$$

Since A is invertible, $a_{11}a_{22}-a_{12}a_{21}$ OR $a_{11}a_{32}-a_{12}a_{31}$ is nonzero. Assuming $a_{11}a_{22}-a_{12}a_{21}$ is nonzero, we get:

$$\begin{bmatrix} a_{11} & a_{12} & a_{13} \\ 0 & a_{11}a_{22}-a_{12}a_{21} & a_{11}a_{23}-a_{13}a_{21} \\ 0 & 0 & a_{11}\Delta \end{bmatrix},$$

where $\Delta = a_{11}a_{22}a_{33} + a_{12}a_{23}a_{31} + a_{13}a_{21}a_{32}$
$\quad - a_{11}a_{23}a_{32} - a_{12}a_{21}a_{33} - a_{13}a_{22}a_{31}$

↑ determinant

$$\Delta = a_{11} \cdot \det\begin{bmatrix} a_{22} & a_{23} \\ a_{32} & a_{33} \end{bmatrix} + a_{12} \cdot \det\begin{bmatrix} a_{21} & a_{23} \\ a_{31} & a_{33} \end{bmatrix} + a_{13} \cdot \det\begin{bmatrix} a_{21} & a_{22} \\ a_{31} & a_{32} \end{bmatrix}$$

*det is determinant of 2×2 matrix

Recursive Definition:

For an $n \times n$ matrix $A = [a_{ij}]$, the determinant is the sum of n terms of the form $\pm a_{1j} \det A_{1j}$ (+/- alternating) and the entries $a_{11}, \dots, a_{1n}$ form the first row of A.

$$\Rightarrow \sum_{j=1}^{n} (-1)^{1+j} a_{1j} \det A_{1j}$$

e.g. det of:

$$A = \begin{bmatrix} 1 & 5 & 0 \\ 2 & 4 & -1 \\ 0 & -2 & 0 \end{bmatrix} \qquad \det A = 1 \cdot \det\begin{bmatrix} 4 & -1 \\ -2 & 0 \end{bmatrix} - 5 \cdot \det\begin{bmatrix} 2 & -1 \\ 0 & 0 \end{bmatrix} + 0 \cdot \det\begin{bmatrix} 2 & 4 \\ 0 & -2 \end{bmatrix}$$

$$= \underbrace{1 \cdot C_{11} - 5C_{12} + 0C_{13}}_{\text{cofactors}}$$

"cofactor expansion across first row of A" $\Rightarrow$ det.

Sign pattern:

$$\begin{bmatrix} + & - & + & \cdot \\ - & + & - & \\ + & - & + & \\ \vdots & & & \ddots \end{bmatrix}$$

determinant calc. works for any row/column cofactor expansion.

<u>Thm.</u> If A is a triangular matrix, then $\det A$ is the product of the entries on the main diagonal of A.

$$\begin{bmatrix} a & * & * & * \\ 0 & b & * & * \\ 0 & 0 & c & * \\ 0 & 0 & 0 & d \end{bmatrix}$$