02/05

Midterm I: Next Friday (regular class time)

No: textbook/calculator/cheatsheet

Yes: Scratch paper

Covers sections lectured in class (Ch. 1-3)

§ 2.2 Inverse

($n \times n$) matrix A. The inverse of A is an $n \times n$ matrix B s.t. $AB = BA = I$

Thm

For $A = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$ the inverse exists if and only if $ad-bc \neq 0$. then

$$A^{-1} = \frac{1}{ad-bc}\begin{pmatrix} d & -b \\ -c & a \end{pmatrix}$$

Check:

$$A^{-1} \cdot A = \frac{1}{ad-bc}\begin{bmatrix} d & -b \\ -c & a \end{bmatrix}\begin{bmatrix} a & b \\ c & d \end{bmatrix} = \begin{bmatrix} ad-bc & bd-ab \\ -ac+ac & -bc+ad \end{bmatrix}$$

* $ad-bc$ called determinunt of A.

Thm ① If A invertible, then so is A^{-1}, and $(A^{-1})^{-1} = A$

② $(AB)^{-1} = B^{-1}A^{-1}$ (caution: order)

③ $(A^T)^{-1} = (A^{-1})^T$

④ (Cancellation Law) If A is invertible,

$AB = AC \Rightarrow B = C$

Reason: $A^{-1}AB = A^{-1}AC$

$\Rightarrow IB = IC$.

Thm If A invertible, then $A\vec{x} = \vec{b}$ has unique solution.

Prf: $A^{-1}A\vec{x} = A^{-1}\vec{b}$

$I\vec{x} = A^{-1}\vec{b}$.

Consequence: An invertible matrix $(n \times n)$
$\Leftrightarrow$ reduced echelon form is exactly I_n.
(Reason: if there is a unique sol, there are no free variables)

Algorithm: A $(n \times n)$. If $B = (\vec{b}_1, \dots \vec{b}_n)$ is the inverse, then $A(\vec{b}_1, \dots, \vec{b}_n) = (\vec{e}_1, \dots, \vec{e}_n)$

This is just:

$$\begin{cases} A_1\vec{b}_1 = \vec{e}_1 \\ \vdots \\ A_n\vec{b}_n = \vec{e}_n \end{cases}$$

want to solve for $\vec{b}_1, \dots \vec{b}_n$.

Augmented Matrix:

$[A \ \ \vec{e}_1]$ → want to get reduced echelon for for each system.

$\vdots$

$[A \ \ \vec{e}_n]$ → Combine system, get $[A \ \ \vec{e}_1, \dots, \vec{e}_n]$

→ $[A \ \ I_n]$ ← solve

Perform row operations on $[A I]$ to get reduced echelon form, in the form $[I \ \ C]$

Then: $A^{-1} = C$.

Reason: Red. echelon form is:

$$[I \ \vec{c}_1, \dots, \vec{c}_n]$$

Then, system $A\vec{b}_i = \vec{e}_i$ has matrix $[A \ \vec{e}_i]$ which becomes $[I \ c_i]$.

This is the system $I_n \vec{b}_i = \vec{c}_i$ which has solution $\vec{b}_i = \vec{c}_i$.

e.x. Find inverse of $A = \begin{bmatrix} 0 & 3 & 4 \\ 1 & 0 & 0 \\ 2 & 5 & 6 \end{bmatrix}$

Row-reduce

$$\left[\begin{array}{ccc|ccc} 0 & 3 & 4 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 & 1 & 0 \\ 2 & 5 & 6 & 0 & 0 & 1 \end{array}\right]$$

$$= \left[\begin{array}{ccc|ccc} 1 & 0 & 0 & 0 & 1 & 0 \\ 0 & 3 & 4 & 1 & 0 & 0 \\ 0 & 5 & 6 & 0 & -2 & 1 \end{array}\right]$$

$$= \left[\begin{array}{ccc|ccc} 1 & 0 & 0 & 0 & 1 & 0 \\ 0 & 1 & 4/3 & 1/3 & 0 & 0 \\ 0 & 0 & -2/3 & -5/3 & -2 & 1 \end{array}\right]$$

$$= \left[\begin{array}{ccc|ccc} 1 & 0 & 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & -3 & -4 & 0 \\ 0 & 0 & -2/3 & -5/3 & -2 & 1 \end{array}\right]$$

A^{-1}.

$$= \left[\begin{array}{ccc|ccc} 1 & 0 & 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & -3 & -4 & 0 \\ 0 & 0 & 1 & 5/2 & 3 & -3/2 \end{array}\right]$$