02/04

<u>Dfn</u> Let $T: V_1 \to V_2$ be a linear trans.
Then, $S: V_2 \to V_1$ is an inverse for T iff.

(i) $T \circ S = id_{V_2}$ → $S(T)$

(ii) $S \circ T = id_{V_1}$ → $T(S)$

<u>Dfn</u> Let A be a matrix (m×n). Then, A is invertible iff $T_A$ is invertible ($T_A: \mathbb{R}^n \to \mathbb{R}^m$, $T_A(x) := Ax$)

<u>Prp</u> A is invertible iff $\exists$ n×m matrix B s.t.

(i) $AB = id_{n \times m}$

(ii) $BA = id_{m \times n}$

$A = \begin{bmatrix} 1 & 0 \end{bmatrix}$, $B = \begin{bmatrix} 1 \\ 0 \end{bmatrix}$

$AB = \begin{bmatrix} 1 \end{bmatrix}$

$BA = \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}$

<u>Gauss-Jordan Elimination</u>

$[A \,\vdots\, Id] \xrightarrow{\text{row reduce}} [Id \,\vdots\, A^{-1}]$

Ex: inverse of $A = \begin{bmatrix} 1 & 3 \\ 2 & 0 \end{bmatrix}$

$$\left[\begin{array}{cc|cc} 1 & 3 & 1 & 0 \\ 2 & 0 & 0 & 1 \end{array}\right]$$

$$\left[\begin{array}{cc|cc} 1 & 3 & 1 & 0 \\ 0 & -6 & -2 & 1 \end{array}\right]$$

$$= \left[\begin{array}{cc|cc} 1 & 0 & 0 & 1/2 \\ 0 & -6 & -2 & 1 \end{array}\right]$$

$$= \left[\begin{array}{cc|cc} 1 & 0 & 0 & 1/2 \\ 0 & 1 & 1/3 & -1/6 \end{array}\right]$$

$Ax = b$ if $A$ is invertible, there exists a unique solution s.t. $x = A^{-1}b$

3 Fundamental Operations w/ Elem. matrices

(i) Row swap $r_3 \leftrightarrow r_2$ elementary matrix: $\begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{bmatrix}$

$A \xrightarrow{r_3 \leftrightarrow r_2} B = E_{r_2 \leftrightarrow r_3} A$

(ii) scale by $\neq 0$ $r_2 \to 3r_2$ $E_{r_2 \to 3r_2} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 3 & 0 \\ 0 & 0 & 1 \end{bmatrix}$

(iii) Add multiple of row to another

$r_1 \to r_1 - 5r_3$ $E_{r_1 \to r_1 - 5r_3} = \begin{bmatrix} 1 & 0 & -5 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$

$$\begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{bmatrix} \rightarrow \begin{matrix} r_1 \text{ new} = 1\cdot r_1 + 0\cdot r_2 + 0\cdot r_3 \\ r_2 \text{ new} = 0\cdot r_1 + 0\cdot r_2 + 1\cdot r_3 \\ r_3 \text{ new} = 0\cdot r_1 + 1\cdot r_2 + 0\cdot r_3 \end{matrix}$$

$$[A \mid id] \xrightarrow{\text{rowops}} [id \mid A^{-1}]$$

$$\underbrace{E_n \ldots E_2 E_1}_{E} [A \mid id]$$

$$= [EA \mid E]$$

$EA = id$ if $A$ invertible $\rightarrow$ $AE = id$

and so $E = A^{-1}$.

Prp $T: V_1 \rightarrow V_2$

$\dim(V_1) = \dim(V_2) < \infty$

Then, $T$ is injective iff $T$ surjective

The following are equivalent (TFAE):

(i) $T$ surjective

(ii) $T$ injective

(iii) $T$ bijective

(iv) $T$ invertible

proof: $\dim(V_2) \overset{\text{by hypothesis}}{=} \dim(V_1) \overset{\text{by Rank Nullity}}{=} \dim(\ker(T)) + \dim(\mathrm{Img}(T))$

Prp $T: V_1 \rightarrow V_2$ is injective iff $\ker(T) = 0$

iff $\dim(\ker(T)) = 0$

$T: V_1 \rightarrow V_2$ is surjective iff $\mathrm{Im}(T) = V_2$

iff $\dim(\mathrm{Im}(T)) = \dim(V_2)$

((i) $\Rightarrow$ (ii)) $\dim(\ker(T)) = 0 \Rightarrow \dim(\mathrm{Im}(T)) = \dim(V_2)$

((ii) $\Rightarrow$ (iii)) $\dim(\mathrm{Img}(T)) = \dim(V_2) \Rightarrow \dim(\ker(T)) = 0$
((iii) $\Rightarrow$ (iv)) Immediate (you can map $x \to y$
$y \to x$ )

Prp $f: X \to Y$ bij iff $f$ invertible
Pf Suppose $f$ is bijective. ie $\forall y \in Y\ \exists!\ x_y \in X : f(x_y) = y$

Define $g: Y \to X$ by $g(y) := x_y$.
$f(g(y)) = y$ Look at $x_{f(x)}$: $f(x_{f(x)}) = f(x)$

$$A = \begin{bmatrix} 1 & 1 & 1 \\ 2 & 1 & 0 \\ 3 & 1 & 1 \end{bmatrix}$$ RREF $[A \,\vdots\, 0]$

$$= \begin{bmatrix} 1 & 1 & 1 \\ 0 & -1 & -2 \\ 0 & -2 & -2 \end{bmatrix}$$

$$= \begin{bmatrix} 1 & 1 & 1 \\ 0 & -1 & -2 \\ 0 & 0 & 2 \end{bmatrix}$$

$$= \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \checkmark$$

$\frac{d}{dx}\ \mathbb{R}[x] \to \mathbb{R}[x]$

$\mathrm{Ker}(\frac{d}{dx}) = \mathrm{Span}(\{1\})$
$\mathrm{Im}(\frac{d}{dx}) = \mathbb{R}[x]$

fail b/c infinite dim.