02/03

## Matrix Multiplication

$$\underset{m \times n}{\underset{\uparrow}{A}}\ \underset{n \times p}{\underset{\uparrow}{B}} = (\underset{\text{1st col}}{A\vec{b}_1} \quad \underset{\text{2nd col}}{A\vec{b}_2} \quad \cdots \quad A\vec{b}_p) \Rightarrow m \times p \text{ result}$$

e.g.

$$\begin{bmatrix} 1 & -1 & 0 \\ 2 & 3 & -2 \end{bmatrix} \times \begin{bmatrix} 1 & 4 \\ 2 & 5 \\ 3 & 6 \end{bmatrix}$$

$$= \left[ \begin{bmatrix} 1 & -1 & 0 \\ 2 & 3 & -2 \end{bmatrix} \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix} \quad \begin{bmatrix} 1 & -1 & 0 \\ 2 & 3 & -2 \end{bmatrix} \begin{bmatrix} 4 \\ 5 \\ 6 \end{bmatrix} \right]$$

$$= \begin{bmatrix} -1 & -1 \\ 2 & 11 \end{bmatrix} \Rightarrow \begin{bmatrix} 1 \cdot 1 - 2 \cdot 1 + 0 \cdot 3 & 1 \cdot 4 - 1 \cdot 5 \\ 2 \cdot 1 + 3 \cdot 2 + -2 \cdot 3 & 2 \cdot 4 + 3 \cdot 5 - 2 \cdot 6 \end{bmatrix}$$

e.g. $\underset{(1 \times n)}{[a_1, a_2, \ldots, a_n]} \underset{(n \times 1)}{\begin{bmatrix} b_1 \\ \vdots \\ b_n \end{bmatrix}}$ $\quad$ $a_i, b_j$ are #s. not rows

$\Rightarrow$ $1 \times 1$ matrix (a number) $\quad a_1 b_1 + a_2 b_2 + \ldots + a_n b_n$

this is called a <u>dot product</u>.

$$\underset{(n \times 1)}{\begin{bmatrix} b_1 \\ \vdots \\ b_n \end{bmatrix}} \times \underset{(1 \times n)}{[a_1, \ldots, a_n]} = \underset{(n \times n)}{\begin{bmatrix} b_1 a_1 & \cdots & b_1 a_n \\ \vdots & & \vdots \\ b_n a_1 & \cdots & b_n a_n \end{bmatrix}}$$

matrix multiplication is <u>not</u> commutative!

<u>Recipe for A B</u> (matrix multiplication)
the $(i,j)$ entry (entry of AB in ith row
is dot product                nth col)
of ith row of A with jth col of B.

$$\begin{bmatrix} 1 & -1 & 0 \\ 2 & 3 & -2 \end{bmatrix} \begin{bmatrix} 1 & 4 \\ 2 & 5 \\ 3 & 6 \end{bmatrix} \Rightarrow \begin{matrix} [1\ -1\ 0] \cdot [1\ 2\ 3] \\ [2\ 3\ -2] \cdot [1\ 2\ 3] \\ [1\ -1\ 0] \cdot [4\ 5\ 6] \\ [2\ 3\ -2] \cdot [4\ 5\ 6] \end{matrix}$$

<u>Matrix Multiplication Laws</u>

① (Distribution) $A(B+C) = AB + AC$

② (Associative) $A(BC) = (AB)C$

③ ( !commutative) not commutative! $AB \neq BA$

<u>e.g</u> $\begin{bmatrix} 2 & 0 \\ 0 & 3 \end{bmatrix} \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix} = \begin{bmatrix} 0 & 2 \\ 0 & 0 \end{bmatrix}$

NOT EQUAL!

$\begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} 2 & 0 \\ 0 & 3 \end{bmatrix} = \begin{bmatrix} 0 & 3 \\ 0 & 0 \end{bmatrix}$

Special Matrices

① zero matrix (every element is 0)

$A + 0 = A$ $\quad\quad B \cdot 0 = 0$

② Vectors in $\mathbb{R}^m$ = $(m \times 1)$ matrix

③ Identity Matrix

$$I_n = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \quad (n \times n \rightarrow n = n)$$

can be written as just $I$.

$$\underset{(m\times n)(n\times n)}{AI} = \underset{(m\times n)}{A} \qquad I_n^n = I_n.$$

$$\underset{(m\times m)(m\times n)}{IA} = \underset{(m\times n)}{A}$$

④ Scalar Matrix

$$c \cdot I_n = \begin{bmatrix} c & 0 & 0 \\ 0 & c & 0 \\ 0 & 0 & c \end{bmatrix}$$

$$(\underbrace{cI_n}_{\text{matrix multiple}})A = \underbrace{cA}_{\text{scalar multiple}}.$$

__Transpose of Matrix__

$A = m \times n$

$A^T = n \times m$ its rows are columns of $A$

e.g. $A = \begin{bmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \end{bmatrix}$

$$A^T = \begin{bmatrix} 1 & 4 \\ 2 & 5 \\ 3 & 6 \end{bmatrix}$$

Properties:

① $(A^T)^T = A$

② $(A+B)^T = A^T + B^T$

③ $(AB)^T = B^T A^T$

§ 2.2 Inverse - only consider square matrix

$A$ $n \times n$ matrix (square)

Say $A$ is invertible if there is a matrix $C$ such that $AC = I$ and $CA = I$. In this case, $C$ is the inverse of $A$ (written as $A^{-1}$).

invertible : nonsingular

not invertible : singular

Remarks: ① The inverse, if it exists, is unique. (determined by $A$).

② Either $AC = I$ or $CA = I$ is sufficient

e.g $\begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix}^{-1} = \begin{bmatrix} 4 & -2 \\ -3 & 1 \end{bmatrix}$

$\hookrightarrow \frac{-1}{2} \begin{bmatrix} 4 & -2 \\ -3 & 1 \end{bmatrix}$