02/02

<u>Prp</u> let $\{v_1, \dots, v_d\} \subseteq V$ be a basis $v \in V$.
Then, $\exists ! \, \alpha_1, \dots, \alpha_d \in \mathbb{R}$ s.t. $v = \alpha_1 v_1 + \alpha_d v_d$.

$$[v]_{\mathcal{B}} = \begin{bmatrix} \alpha_1 \\ \vdots \\ \alpha_d \end{bmatrix}$$

coordinates

dimension

$V := \{ y \in C^{\infty}(\mathbb{R}) : y'' + y = 0 \}$

$C : \{ e^{ix}, e^{-ix} \}$ is a basis for V, $\cos(x) \in V$,

$$[\cos(x)]_{\mathcal{B}} = \begin{bmatrix} \alpha_1 \\ \alpha_2 \end{bmatrix} = \begin{bmatrix} 1/2 \\ 1/2 \end{bmatrix}$$

$e^{ix} = \cos(x) + i \sin(x)$ ← Euler's formula

$e^{-ix} = \cos(x) - i \sin(x)$

$$\sin(x) = \frac{1}{2i} e^{ix} - \frac{1}{2i} e^{ix}$$

$$[\sin(x)]_e = \begin{bmatrix} 1/2i \\ -1/2i \end{bmatrix}$$

$$[e^{ix}]_e = \begin{bmatrix} 1 \\ 0 \end{bmatrix}, \quad [e^{-ix}] = \begin{bmatrix} 0 \\ 1 \end{bmatrix}$$

$V := \mathbb{R}^3$

$\mathcal{B} := \left\{ \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} \right\}$ ← standard basis

$\mathcal{C} := \left\{ \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}, \begin{bmatrix} 1 \\ -1 \\ 1 \end{bmatrix} \right\}$

$$[v]_{\mathcal{B}} = \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix}$$

$$[v]_{\mathcal{C}} \quad \left[\begin{array}{cccc|c} 1 & 1 & 1 & & 1 \\ 0 & 1 & -1 & & 2 \\ 0 & 1 & 1 & & 3 \end{array}\right]$$

Coordinates of Linear Transformations

Prop let $T: V_1 \to W$ be a lin. trans

$\mathcal{B} = \{b_1, \dots, b_d\}$ basis for V.

$\mathcal{C} = \{c_1, \dots, c_e\}$ basis for W.

Then, $\exists!$ $e \times d$ matrix $[T]_{\mathcal{C} \leftarrow \mathcal{B}}$ s.t.

$$[T(v)]_{\mathcal{C}} = [T]_{\mathcal{C} \leftarrow \mathcal{B}} [v]_{\mathcal{B}} \quad \forall\, v \in V$$

Furthermore,

$$[T]_{\mathcal{C} \leftarrow \mathcal{B}} = \left[[T(b_1)]_{\mathcal{C}}, \dots, [T(b_d)]_{\mathcal{C}} \right]$$

$id_V : V \to V$

$\mathcal{B}, \mathcal{C}$ diff. bases for V

$[id_V]_{\mathcal{C} \leftarrow \mathcal{B}}$ change of basis matrix

$[v]_{\mathcal{C}} = [id]_{\mathcal{C} \leftarrow \mathcal{B}} [v]_{\mathcal{B}}$

$$[T \circ S]_{\mathcal{D} \leftarrow \mathcal{B}} = [T]_{\mathcal{D} \leftarrow \mathcal{C}} [S]_{\mathcal{C} \leftarrow \mathcal{B}}$$

$V := \{ y \in C^{\infty}(\mathbb{R}) : y'' + y = 0 \}$

$T : V \to V$, $T(f) = f'$

$\mathcal{B} := \{\cos(x), \sin(x)\}$, $\mathcal{C} = \{e^{ix}, e^{-ix}\}$

We can calculate:

$$\left[\frac{d}{dx}\right]_{\mathcal{B} \leftarrow \mathcal{B}} \overset{\left[[-\sin(x)]_{\mathcal{B}}, [\cos(x)]_{\mathcal{B}} \right]}{=} \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix}$$

$$\left[\frac{d}{dx}\right]_{\mathcal{C} \leftarrow \mathcal{B}} = \left[\left[\frac{d}{dx}\cos(x)\right]_{\mathcal{C}}, \left[\frac{d}{dx}\sin(x)\right]_{\mathcal{C}} \right] = \left[\begin{bmatrix} -\frac{1}{2}i \\ \frac{1}{2}i \end{bmatrix}, \begin{bmatrix} \frac{1}{2} \\ \frac{1}{2} \end{bmatrix} \right]$$

$$\left[\frac{d}{dx}\right]_{\mathcal{B} \leftarrow \mathcal{C}} \overset{\left[[ie^{ix}], [-ie^{-ix}] \right]}{=} \begin{bmatrix} i & -i \\ -1 & 1 \end{bmatrix}$$

$$\left[\frac{d}{dx}\right]_{\mathcal{C} \leftarrow \mathcal{C}} \overset{\left[[ie^{ix}], [-ie^{-ix}] \right]}{=} \begin{bmatrix} i & 0 \\ 0 & -i \end{bmatrix}$$

§ 1.9

$T: \mathbb{R}^2 \to \mathbb{R}^4$. Standard Matrix: Basis$_3$ of $\left\{\begin{bmatrix}1\\0\\0\end{bmatrix}, \begin{bmatrix}0\\1\\0\end{bmatrix}, \begin{bmatrix}0\\0\\1\end{bmatrix}\right\}$

$$T(e_1) = \begin{bmatrix}-3\\1\\3\\1\end{bmatrix}, \quad T(e_2) = \begin{bmatrix}5\\-2\\0\\0\end{bmatrix}$$

$$[T]_{B_4 \leftarrow B_2} \left[[T(e_1)]_{B_4}, [T(e_2)]_{B_4} \right]$$

$$= \left[\begin{bmatrix}-3\\1\\3\\1\end{bmatrix}, \begin{bmatrix}5\\-2\\0\\0\end{bmatrix} \right].$$

$T: \mathbb{R}^2 \to \mathbb{R}^2$, rotation by θ

$$[T] = \begin{bmatrix}\cos(\theta) & -\sin(\theta)\\ \sin(\theta) & \cos(\theta)\end{bmatrix}$$

$T: \mathbb{R}^2 \to \mathbb{R}^2$

$T(e_1) = e_1, \quad T(e_2) = e_2 + 2e_1$

$$[T] = \begin{bmatrix}1 & 2\\ 0 & 1\end{bmatrix}$$

$\mathbb{R}[x]/(x^3)$ - polynomials of deg at most 2

$\uparrow$ set $x^3 = 0$. Basis: $\{1, x, x^2\}$

$\mathbb{R}[x] \leftarrow$ all polynomials