02/01/16

e.g. $\vec{v}_1$ is linearly dep. $\overset{\text{iff}}{\Leftrightarrow}$ $\vec{v}_1 = 0$

test: $\vec{v}_1$ is linearly dep iff $x_1 \vec{v}_1 = \vec{0}$ has a nonzero solution

§1.9 Matrix of Linear Transformation

Recall: A linear transformation $T: \mathbb{R}^m \to \mathbb{R}^n$ is a map satisfying

① $T(\vec{x}+\vec{y}) = T(\vec{x}) + T(\vec{y})$

② $T(c\vec{x}) = cT(\vec{x})$

$\Rightarrow T(c_1\vec{v}_1 + c_2\vec{v}_2 \dots c_r\vec{v}_r) = T(c_1\vec{v}_1) + T(c_2\vec{v}_2) + \dots + T(c_r\vec{v}_r)$

Also have matrix transformation:

$T: \mathbb{R}^n \to \mathbb{R}^m$

$T(\vec{x}) = A\vec{x}$, where A is an $m \times n$ matrix

Thm: Any linear transformation is a matrix transformation, and the matrix A is unique.

e.g. $T: \mathbb{R}^2 \to \mathbb{R}^3$

$T\begin{pmatrix}1\\0\end{pmatrix} = \begin{bmatrix}2\\3\\4\end{bmatrix}$

$T\begin{pmatrix}0\\1\end{pmatrix} = \begin{bmatrix}-2\\-1\\0\end{bmatrix}$

then, $T\begin{pmatrix}x_1\\x_2\end{pmatrix} = T\left(x_1\begin{bmatrix}1\\0\end{bmatrix} + x_2\begin{bmatrix}0\\1\end{bmatrix}\right)$

$= x_1 T\left(\begin{bmatrix}1\\0\end{bmatrix}\right) + x_2 T\left(\begin{bmatrix}0\\1\end{bmatrix}\right)$

$= x_1\begin{bmatrix}2\\3\\4\end{bmatrix} + x_2\begin{bmatrix}-2\\-1\\0\end{bmatrix}$

$= \begin{bmatrix}2x_1 - 2x_2\\3x_1 - x_2\\4x_1\end{bmatrix}$

$$\begin{bmatrix} 2x_1 & -2x_2 \\ 3x_1 & -x_2 \\ 4x_1 & \end{bmatrix} = \underbrace{\begin{bmatrix} 2 & -2 \\ 3 & -1 \\ 4 & 0 \end{bmatrix}}_{A} \underbrace{\begin{bmatrix} x_1 \\ x_2 \end{bmatrix}}_{\vec{x}}$$

General Result

$T: \mathbb{R}^n \to \mathbb{R}^m$

$$\vec{e}_1 = \begin{bmatrix} 1 \\ 0 \\ \vdots \\ 0 \end{bmatrix}, \vec{e}_2 = \begin{bmatrix} 0 \\ 1 \\ \vdots \\ 0 \end{bmatrix}, \dots, \vec{e}_n = \begin{bmatrix} 0 \\ 0 \\ \vdots \\ 1 \end{bmatrix}$$

then T is ~~represented~~ by the matrix

$$\begin{array}{c} \begin{bmatrix} T(\vec{e}_1), T(\vec{e}_2), \dots, T(\vec{e}_n) \end{bmatrix} \\ \wedge \qquad\qquad\qquad \wedge \\ \text{column} \qquad\qquad \text{column} \end{array}$$ ← standard matrix

examples

① $T(\vec{x}) = \vec{x}$ $(\mathbb{R}^m \to \mathbb{R}^n)$

$$T\left(\begin{bmatrix} 0 \\ 0 \\ \vdots \\ 1 \\ 0 \end{bmatrix}\right) = T\left(\begin{bmatrix} 0 \\ 0 \\ \vdots \\ 1 \\ 0 \end{bmatrix}\right)$$ ← doesn't change

Matrix:

$$[\vec{e}_1, \vec{e}_2, \dots, \vec{e}_n] = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ \vdots & \vdots & \vdots & \vdots \\ 0 & 0 & 0 & 1 \end{bmatrix}$$ ← identity matrix I_n

↑ size $(n \times n)$

diagonal

§ 2.1 Matrix Operations

* $m \times n$ matrix is square if $m = n$.

① Addition. $A+B$ = sum of corresponding entries

- A, B need to be same size

e.g. $\begin{bmatrix} a & b \\ c & d \end{bmatrix} + \begin{bmatrix} e & f \\ g & h \end{bmatrix} = \begin{bmatrix} a+e & b+f \\ c+g & d+h \end{bmatrix}$

② Scalar multiplication:

$c \cdot A$ = $c \cdot$ entries of A.

eg $2 \begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix} = \begin{bmatrix} 2 & 4 \\ 6 & 8 \end{bmatrix}$

③ Matrix Multiplication

$$\underset{(m \times n)}{A} \cdot \underset{(n \times p)}{B} = \underset{(m \times p)}{\text{matrix}}$$

same

Law:

$B = [\vec{b}_1, \ldots, \vec{b}_p]$

$A \cdot B = [A\vec{b}_1, A\vec{b}_2, \ldots, A\vec{b}_p]$.

② $T(\vec{x}) = c \cdot \vec{x}$

matrix is $c \cdot I_n$ ← scalar matrix

③ $T\left(\begin{bmatrix} x \\ y \end{bmatrix}\right) = T\left(\begin{bmatrix} y \\ x \end{bmatrix}\right)$

reflection with respect to line $y = x$

Matrix: $\begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}$

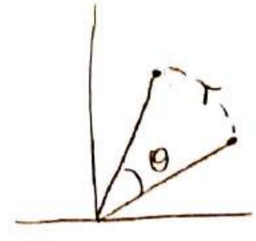
④ $T\left(\begin{bmatrix} x \\ y \end{bmatrix}\right) = \begin{bmatrix} -x \\ y \end{bmatrix}$

reflection with respect to line $y=0$.

$$\begin{bmatrix} -1 & 0 \\ 0 & 1 \end{bmatrix}$$

⑤ Rotation with angle θ

Matrix: $\begin{bmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix}$



$$= \begin{bmatrix} x\cos\theta - y\sin\theta \\ x\sin\theta + y\cos\theta \end{bmatrix}$$