01/28

Linear independence : $\{\vec{v}_0, \dots, \vec{v}_n\} \subseteq V$ is lin. ind. iff no vector in $\{v_1, \dots, v_n\}$ can be written as a linear comb. of the others.

<u>Prp</u> $\{v_1, \dots, v_n\}$ is lin. dep. ~~iff~~ when:
$$\alpha_1 v_1 + \dots + \alpha_n v_n = 0 \not\Rightarrow \alpha_1 = 0, \dots, \alpha_n = 0$$

<u>Prp</u> $\{v_1, \dots v_n\}$ is linearly dep. iff $\exists$ constants $\alpha_1, \dots, \alpha_n$ not all $= 0$ s.t. $\alpha_1 \vec{v}_1 + \dots + \alpha_n \vec{v}_n = \vec{0}$.

$\{\cos(x), \sin(x)\} \subseteq C^\infty(\mathbb{R})\}$ is lin. ind.?
Suppose $\alpha_1 \cos(x) + \alpha_2 \sin(x) = 0$
WTS (want to show) $\alpha_1 = 0 = \alpha_2$
Plug in $x=0$: $\alpha_1 \cdot 1 + \alpha_2 \cdot 0 = 0 \Rightarrow \alpha_1 = 0$
Plug in $x = \pi/2$: $\alpha_1 \cdot 0 + \alpha_2 \cdot 1 = 0 \Rightarrow \alpha_2 = 0$

$\begin{bmatrix} 1 \\ 2 \\ 3 \\ 4 \end{bmatrix}, \begin{bmatrix} 1 \\ 0 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 1 \\ 1 \\ 1 \\ 1 \end{bmatrix} \to$ linearly ind?

Suppose
$$\alpha_1 \begin{bmatrix} 1 \\ 2 \\ 3 \\ 4 \end{bmatrix} + \alpha_2 \begin{bmatrix} 1 \\ 0 \\ 1 \\ 0 \end{bmatrix} + \alpha_3 \begin{bmatrix} 1 \\ 1 \\ 1 \\ 1 \end{bmatrix} = 0$$

$$\begin{bmatrix} 1 & 2 & 3 & 4 & 1 & 0 \\ 1 & 0 & 1 & 0 & 1 & 0 \\ 1 & 1 & 1 & 1 & 1 & 0 \end{bmatrix} \to \begin{bmatrix} 1 & 0 & 1 & 0 & 1 & 0 \\ 0 & 1 & 0 & 1 & 1 & 0 \\ 0 & 0 & 2 & 2 & 1 & 0 \end{bmatrix}$$

$$= \begin{bmatrix} 1 & 0 & 1 & 0 & 1 & 0 \\ 0 & 1 & 0 & 1 & 1 & 0 \\ 0 & 2 & 2 & 4 & 1 & 0 \end{bmatrix}$$

$$\begin{bmatrix} 1 & 1 & 1 \\ 2 & 0 & 1 \\ 3 & 1 & 1 \\ 4 & 0 & 1 \end{bmatrix}$$

$$= \begin{bmatrix} 1 & 1 & 1 \\ 0 & -2 & -1 \\ 0 & -2 & -2 \\ 0 & -4 & -3 \end{bmatrix}$$

$$= \begin{bmatrix} 1 & 1 & 1 \\ 0 & -2 & -1 \\ 0 & 0 & -1 \\ 0 & 0 & -1 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \end{bmatrix} \qquad \alpha_1, \alpha_2, \alpha_3 = 0.$$

Dfn: $\{v_1, \ldots, v_n\} \subseteq V$ spans V iff $\mathrm{Span}(\{v_1, \ldots, v_n\}) = V$.

Dfn: $\mathcal{B} := \{v_1, \ldots, v_n\} \subseteq$ is a basis iff

(i) $\mathcal{B}$ spans V

(ii) $\mathcal{B}$ is linearly independent

Thm $\mathcal{B}_1, \mathcal{B}_2$ bases of $V \Rightarrow |\mathcal{B}_1| = |\mathcal{B}_2|$ (num of elements)

Dfn: $\dim(V) =$ # of elements in any basis

Thm Rank-Nullity Theorem: has a dimension.

Let $T: V_1 \to V_2$ between finite dim. V spaces.

Then,

$$\dim(V_1) = \dim(\ker(T)) + \dim(\mathrm{Img}(T))$$

$\mathrm{Rank}(T) := \dim(\mathrm{Img}(T))$

$T_A(x) := Ax$, A $m \times n$ matrix

n: (# free var) + (# pivots) *every col. is a free var or pivot.

$\qquad \Downarrow$

$\dim(\ker(T_A)) + \dim(\mathrm{Img}(T_A))$

A is an $m \times n$ matrix, m pivots.

Is $x \overset{\text{map}}{\longmapsto} Ax$ one-to-one?

$n = \dim(\ker(T_A)) + m$

$\dim(\ker(T_A)) = \underbrace{n-m}_{\text{not necessarily } 0,\ m=n.}$

(i.e. 1×2 matrix)

<u>Dfn</u> $f: X \to Y$ is injective (one-to-one) iff $\forall y \in Y$, $\exists$ at most one $x \in X$ s.t. $f(x) = y$.

<u>Dfn</u> $f: X \to Y$ is surjective (onto) iff $\forall y \in Y$ $\exists$ at least one $x \in X$ s.t. $f(x) = y$.

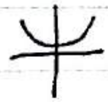
<u>Dfn</u> $f: X \to Y$ is bijective (one-to-one correspondence) iff $\forall y \in Y$ $\exists$ exactly one $x \in X$ s.t. $f(x) = y$.

<u>Prp</u> $f: X \to Y$ is injective iff whenever $f(x) = f(y) \Rightarrow x = y$, iff $x \neq y \Rightarrow f(x) \neq f(y)$

★ <u>Prp</u> $f: X \to Y$ is surjective iff $\mathrm{Im}(f) = Y$.

<u>Ex</u> $x \mapsto x^2$. $f: [0, \infty) \to \mathbb{R} \Rightarrow$ not surj. Johnny: $f: [0,\infty) \to [0,\infty) \Rightarrow$ surj. "I'm a dick."

surjectivity depends on domain & codomain!

$f: \mathbb{R} \to \mathbb{R}$ 

a function can be injective AND surjective

$f: \mathbb{R} \to [0, \infty)$ 

<u>Prp</u> $T_1 : V_1 \to V_2$ is injective iff
$T(x) = 0 \Rightarrow x = 0$ iff $\ker(T) = 0$.

$A_{6\times5}$, $T_A : \mathbb{R}5 \to \mathbb{R}^6$ T_A is NOT onto?
TRUE, $5 = \dim(\ker) + \dim(\mathrm{Img})$
$T: V_1 \to V_2$ only if $\mathrm{Img}(T) = \dim(V_2)$
$\dim(\mathrm{Img}(T)) = 5 - \dim(\ker(T)) \Rightarrow \dim(\mathrm{Img}(T)) \le 5$