§1.4 Matrix Equations

$A\vec{x} = \vec{b}$ — $\vec{b}$: $m \times 1$ column vector in $\mathbb{R}^m$

A: $m \times n$ matrix

$\vec{x}$: $n \times 1$ column vector in $\mathbb{R}^n$

* $m \times n$: m rows, n columns

e.g.
$$\begin{cases} x_1 + 2x_2 + 3x_3 = 4 \\ \quad\;\; x_2 - x_3 = -5 \end{cases}$$

$$= \underbrace{\begin{bmatrix} 1 & 2 & 3 \\ 0 & 1 & -1 \end{bmatrix}}_{A} \underbrace{\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}}_{\vec{x}} = \underbrace{\begin{bmatrix} 4 \\ -5 \end{bmatrix}}_{\vec{b}}$$

If $A = (\vec{a}_1, \dots, \vec{a}_n)$, $\vec{x} = \begin{bmatrix} x_1 \\ \vdots \\ x_n \end{bmatrix}$

then, $A\vec{x} := x_1\vec{a}_1 + \dots + x_n\vec{a}_n$

aka a linear combination of column vectors of A

(its in $\mathbb{R}^n$)

Remark: $(m \times n) \cdot (n \times 1) = (m \times 1)$

somehow, this cancels

e.g.
$$\begin{bmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \\ 7 & 8 & 9 \\ 10 & 11 & 12 \end{bmatrix} \begin{bmatrix} 1 \\ 0 \\ 2 \end{bmatrix} = 1\begin{bmatrix} 1 \\ 4 \\ 7 \\ 10 \end{bmatrix} + 0\begin{bmatrix} 2 \\ 5 \\ 8 \\ 11 \end{bmatrix} + 2\begin{bmatrix} 3 \\ 6 \\ 9 \\ 12 \end{bmatrix}$$

$$= \begin{bmatrix} 7 \\ 16 \\ 25 \\ 34 \end{bmatrix} = \text{"}\vec{b}\text{"}$$

Consider the eqn:

$A\vec{x} = \vec{b}$ A: $m \times n$ matrix $\vec{b} = \begin{bmatrix} b \\ \\ b_m \end{bmatrix}$

$\vec{x} = \begin{bmatrix} x_1 \\ \vdots \\ x_n \end{bmatrix}$

Any system can be written in this form.

$A\vec{x} = \vec{b}$ has a solution $\Leftrightarrow$ $\vec{b}$ is a linear comb of column vectors of A

Theorem: The following are equivalent:

(1) $A\vec{x} = \vec{b}$ has a solution for every $\vec{b}$.

(2) Every $\vec{b}$ is a linear comb. $\vec{b}$ of column vectors of A.

(3) Column vectors of A span $\mathbb{R}^m$

(4) A has a pivot entry in every rows.

Reason (4) $\Rightarrow$ (1)

Consider eqn $A\vec{x} = \vec{b}$.

Augmented matrix:

$$\underbrace{\begin{bmatrix} \underset{m\times n}{A} & \underset{m\times 1}{\vec{b}} \end{bmatrix}}_{m\times(n+1)}$$

$\rightarrow$ Using elementary row operations get reduced echelon form

$$\begin{bmatrix} A' & \vec{b}' \end{bmatrix}$$

Then A' is a reduced echelon form for A.

Then (4) says A' has pivot in every row.

Then $[A' \vec{b}']$ looks like

$$\left[\begin{array}{cccc|c} 1 & \cdots & * & & \vec{b}' \\ 0 & 0 & 1 & & \vec{b}'_m \end{array}\right]$$

Also, there is no $0\,0 \dots 0\,0$ in A'.

$\rightarrow$ so system has solutions.

§1.5 Solution Sets

Homogenous system:

$A\vec{x} = \vec{0}$.

Then, $\vec{x} = \begin{bmatrix} 0 \\ \vdots \\ 0 \end{bmatrix}$ is always a sol called a trivial sol.

Other solutions (if any) are called nontrivial sol.

ex $\left[\begin{array}{cccccc|c} 0 & \triangle\!\!\!1 & 0 & 0 & 2 & & 0 \\ 0 & 0 & 0 & \triangle\!\!\!1 & 0 & & 0 \end{array}\right]$

free variables

Solution set: $\vec{x} = \begin{bmatrix} x_1 \\ -2x_5 \\ x_3 \\ 0 \\ x_5 \end{bmatrix}$, $x_1 \in \mathbb{R}$, $x_3 \in \mathbb{R}$, $x_5 \in \mathbb{R}$

$= \left\{ x_1 \begin{bmatrix} 1 \\ 0 \\ 0 \\ 0 \\ 0 \end{bmatrix} + x_3 \begin{bmatrix} 0 \\ 0 \\ 1 \\ 0 \\ 0 \end{bmatrix} + x_5 \begin{bmatrix} 0 \\ -2 \\ 0 \\ 0 \\ 1 \end{bmatrix} \right\}$

$= \text{Span} \left\{ \begin{bmatrix} 1 \\ 0 \\ 0 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 0 \\ 1 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ -2 \\ 0 \\ 0 \\ 1 \end{bmatrix} \right\}$