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A: $m \times n$ matrix

$T_A : \mathbb{R}^n \to \mathbb{R}^m, \quad T_A(x_n) := Ax$

$\mathrm{Kern}(T_A) := \{x \in \mathbb{R}^n : T_A(x) = 0\}$

$\qquad\qquad = \{x \in \mathbb{R}^n : Ax = 0\}$

Linear Trans.	vs	Matrices	
(kernal) T_A	$\leftarrow\!\shortmid$	A	(null space)
(Img) T	$\mapsto$	$[T]_{C \leftarrow B}$	(columnspace)

$T : V_1 \to V_2$

$\mathrm{Ker}(T) := \{x \in V_1 : T(x) = 0\} \subseteq V_1.$

$\mathrm{Img}(T) := \{T(x) \in V_2 : x \in V_1\} \subseteq V_2$

Rank-Nullity Thm:

$\dim(V_1) = \dim(\mathrm{Ker}(T)) + \dim(\mathrm{Im}(T))$

If $T = T_A$, $n = (\#\text{free var}) + (\#\text{pivots})$

$A = \begin{bmatrix} 0 & 0 \\ 0 & 0 \\ 0 & 0 \end{bmatrix} \qquad T_A : \mathbb{R}^2 \to \mathbb{R}^3$

$\mathrm{Im}(T_A) = \{Ax : x \in \mathbb{R}^2\}$

$\qquad = \{0\}$

$T : V_1 \to V_2$ is a linear transformation if:

(i) $T(x+y) = T(x) + T(y)$

(ii) $T(cx) = cT(x)$

Dfn: $S \subseteq V$ is a subspace if S is a v-space.

Subspace if:

(i) $0 \in S$

(ii) $v \in S \Rightarrow cv \in S$

(iii) $v, w \in S \Rightarrow v+w \in S$

$$Ax = \begin{bmatrix} 1 & 0 \\ 0 & 0 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} \qquad T_A : \mathbb{R}^2 \to \mathbb{R}^3$$

$$\begin{aligned} Img(T_A) &= \{ Ax : x \in \mathbb{R}^2 \} \\ &= \{ \begin{bmatrix} x_1 \\ 0 \\ 0 \end{bmatrix}, x_1 \in \mathbb{R} \} \\ &= span(\{ \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} \}) \end{aligned}$$

$Ax = b \leftarrow Im(T_A)$ set of all x for which there is some solution

$Im(T_A) = \{ b \in \mathbb{R}^m : Ax = b$ has some sln $\}$
($Ax = b$ is consistent)

$$\begin{aligned} Im(T_A) &:= \{ T_A(x) \in \mathbb{R}^m : x \in \mathbb{R}^n \} \\ &:= \{ Ax \in \mathbb{R}^m : x \in \mathbb{R}^n \} \end{aligned}$$

$$v_4 = \begin{bmatrix} -5 \\ -2 \\ 6 \end{bmatrix} \qquad v_1 = \begin{bmatrix} 0 \\ 1 \\ 3 \end{bmatrix} \qquad v_2 = \begin{bmatrix} 1 \\ 3 \\ 7 \end{bmatrix} \qquad v_3 = \begin{bmatrix} 4 \\ 5 \\ 7 \end{bmatrix}$$

No, v_4 not in span.

$v_4 \overset{?}{\in} \text{Span}(\{v_1, v_2, v_3\})$; $A := [v_1\ v_2\ v_3]$

$v_4 \overset{?}{\in} \text{Img}(T_A)$

$v_1 : \begin{bmatrix} 0 \\ 1 \\ 3 \end{bmatrix} \quad v_2 \begin{bmatrix} 1 \\ 3 \\ 7 \end{bmatrix} \quad v_3 \begin{bmatrix} 4 \\ 5 \\ 7 \end{bmatrix}$

$A \sim \begin{bmatrix} 1 & 0 & -7 \\ 0 & 1 & 4 \\ 0 & 0 & 0 \end{bmatrix}$

Img given by span of pivot columns

$$\left[\begin{array}{ccc|c} 0 & 1 & 4 & 6 \\ 1 & 3 & 5 & 0 \\ 3 & 7 & 7 & 0 \end{array}\right] \Rightarrow \left[\begin{array}{ccc|c} 0 & 1 & 4 & 6 \\ 1 & 3 & 5 & 0 \\ 0 & -2 & -8 & 0 \end{array}\right]$$

$\text{Ker}(T_A) = \left\{ x_3 \begin{bmatrix} 7 \\ -4 \\ 1 \end{bmatrix} : x_3 \in \mathbb{R} \right\}$ kernal ∝ free variables

$= \text{Span}\left(\left\{ \begin{bmatrix} 7 \\ -4 \\ 1 \end{bmatrix} \right\}\right)$