12/01/15

16.9 Divergence Theorem

Flashback:

16.5, used Green's Theorem to derive:

$$\int_C \vec{F}\cdot d\vec{n}\, ds = \iint_D \operatorname{div} \vec{F}\, dA.$$

Overview:

The divergence theorem is a higher-dimensional analog of. Recall how solid regions of type 1, 2 and 3 in 15.7 are defined:

Type 1: $E = \{(x,y,z) \mid (x,y) \in D,\ z_1(x,y) \le z \le z_2(x,y)\}$

Type 2: " $(y,z) \in D,\ x_1(y,z) \le x \le x_2(y,z)\}$

Type 3: " $(z,x) \in D,\ y_1(x,z) \le y \le y_2(x,z)\}$

Solids simultaneously of types 1, 2, and 3 are <u>simple</u>.

The Divergience theorem:

Let E be a simple sided solid region w/ boundary surface S and has positive orientation.

Let $\vec{F}$ be a vector field on $\mathbb{R}^3$ whose components have cont. partial deriv. on an open region containg E.

Then,

$$\boxed{\underbrace{\iint_S \vec{F}\cdot d\vec{S}}_{\text{flux of } \vec{F} \text{ across } S = \partial E} = \iiint_E \operatorname{div}\vec{F}\, dV.}$$

Proof:

If $\vec{F} = P\hat{i} + Q\hat{j} + R\hat{k}$, then

$$\operatorname{div}\vec{F} = \frac{\partial P}{\partial x} + \frac{\partial Q}{\partial y} + \frac{\partial R}{\partial z}$$

so $$\iiint_E \operatorname{div}\vec{F}\, dV = \iiint_E \frac{\partial P}{\partial x}\, dV + \iiint_E \frac{\partial Q}{\partial y}\, dV + \iiint_E \frac{\partial R}{\partial z}\, dV.$$

On the other hand,

$$\iint_S \vec{F}\cdot d\vec{S} = \iint_S \vec{F}\cdot\vec{n}\, dS = \iint_S (P\hat{i} + Q\hat{j} + R\hat{k})\cdot\vec{n}\, dS$$

$$= \iint_S P\hat{i}\cdot\vec{n}\, dS + \iint_S Q\hat{j}\cdot\vec{n}\, dS + \iint_S R\hat{k}\cdot\vec{n}\, dS.$$

So we need to prove:

- $\iint_S P\hat{i}\cdot\vec{n}\, dS = \iiint_E \frac{\partial P}{\partial x}\, dV$
- $\iint_S Q\hat{j}\cdot\vec{n}\, dS = \iiint_E \frac{\partial Q}{\partial y}\, dV$
- $\iint_S R\hat{k}\cdot\vec{n}\, dS = \iiint_E \frac{\partial R}{\partial z}\, dV$ ← proving this one

Since E is a type 1 region, $E = \{(x,y,z) \mid (x,y)\in D,\ u_1(x,y) \le z \le u_2(x,y)\}$

D = projection of E on x-y plane.

From section 15.7 (Triple Integrals), we have:

$\iiint_E \frac{\partial R}{\partial z}\, dV = \iint_D \left[\int_{u_1(x,y)}^{u_2(x,y)} \partial R/\partial z\,(x,y,z)\right] dz$ because $\frac{\partial R}{\partial z}$ is func. of 3 var.

In the inner integral, x and y are treated like constants.

By FTC, $\int_{u_1(x,y)}^{u_2(x,y)} \frac{\partial R}{\partial z}(x,y,z)\, dz = R(x,y,u_2(x,y)) - R(x,y,u_1(x,y))$.

* So $\iiint_E \frac{\partial R}{\partial z}\, dV = \iint_D [R(x,y,u_2(x,y)) - R(x,y,u_1(x,y))]\, dA$.

$\iint_S R\hat{k}\cdot\vec{n}\, dS$:

S_1: $z = u_1(x,y)$

S_2: $z = u_2(x,y)$

S_3: between S_1 & S_2. "lateral" surface

So $\iint_S R\hat{k}\cdot\vec{n}\, dS = \iint_{S_1} R\hat{k}\cdot\vec{n}\, dS + \iint_{S_2} R\hat{k}\cdot\vec{n}\, dS + \cancelto{0}{\iint_{S_3} R\hat{k}\cdot\vec{n}\, dS}.$

On S_3, $\vec{n}$ is horizontal, so $\vec{n}\cdot\hat{k} = 0$.

On S_2, $\vec{n}$ points upward.

On S_1, $\vec{n}$ points downward.

Using $\iint_S \vec{F}\cdot d\vec{S} = \iint_S \vec{F}\cdot\vec{n}\, dS = \iint_D \left(-P\frac{\partial g}{\partial x} - Q\frac{\partial g}{\partial y} + R\right) dA$

where $\vec{F} = \langle 0, 0, R\rangle$ and $u_2 = g$ we get:

$$\iint_{S_2} R\hat{k}\cdot\vec{n}\, dS = \iint_D R\, dA = \iint_D R(x, y, u_2(x,y))\, dA$$

when $u_1 = g$, we get:

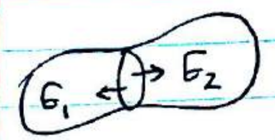
$$\iint_{S_1} R\hat{k}\cdot\vec{n}\, dS = \iint_D R\, dA = -\iint R(x, y, u_1(x,y))\, dA.$$

Combing, we get:

$$\iint_S R\cdot k\cdot\vec{n}\, dS = \iint_D [R(x,y, u_2(x,y)) - R(x,y,u_1(x,y))]\, dA \checkmark.$$

$$= \iiint_E \partial R/\partial z\, dV.$$

Extension to more general Regions:



Only the outer surface "survives" so

$$\sum_{i=1}^{n} \iiint_{E_i} \operatorname{div}\vec{F}\, dV = \iiint_E \operatorname{div}\vec{F}\, dV.$$

E

You can "add" regions.

$E_2 = E - E_1$.

$$\iint_{E_2} \vec{F}\cdot d\vec{S} = \iint_E \vec{F}\cdot d\vec{S} - \iint_{E_1} \vec{F}\cdot d\vec{S}.$$

Div and Flux: Sources and Sinks

Let $\operatorname{div}\vec{F}$ be continuous and $P_0 = (x_0, y_0, z_0)$

Let B_a = ball w/ radius a and center P_0.

Let S_a = boundary of B_a.

If a is small, $\iint_{S_a} \vec{F}\cdot d\vec{S} = \iiint_{B_a} \operatorname{div}\vec{F}\, dV \approx \iiint_{B_a} \operatorname{div}\vec{F}(P_0)\, dV.$

$$= \operatorname{div}\vec{F}(P_0)\cdot V(B_a) = \boxed{\operatorname{div}\vec{F}(P_0)\cdot \tfrac{4}{3}\pi a^3}.$$

gets better as $a \to 0$.

So, div $\vec{F}(P_0)$ is the rate of outward flux per unit volume at P_0.

If div $\vec{F}(P_0) > 0$, the net flow is outward near P_0, and P_0 is called a source.

If div $\vec{F}(P_0) < 0$, the net flow is inward near P_0, and P_0 is called a sink.