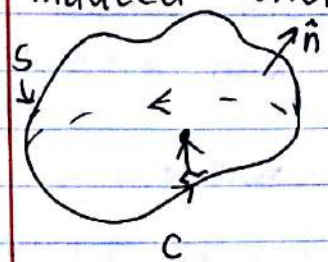
11/31/15

# 16.8 Stoke's Theorem

Induced orientation of Boundary Curves:




- S has a simple closed boundary curve C. The orientation of S induces a natural positive orientation of C.

Imagine you are walking along C with your head in the direction of $\hat{n}$. When you walk in the positive direction of C, S will be on your left.

Overview:

- Stoke's theorem is another analog of FTC #2.
- Stoke's theorem is Green's theorem + another dimension.
  $\hookrightarrow \int_c \vec{F}\cdot d\vec{r} = \iint_D \left(\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y}\right) dA$
  - Green's theorem is a special case of Stoke's theorem

Proof (similar to Green's):

- We prove the "simple", special case.
- Partition more complex surfaces into smaller simple cases.
- curl $\vec{F} \propto$ "circulation" (to be explained.)

Stoke's Theorem:

- Let S be an oriented piecewise smooth surface bounded by a simple p.w. smooth boundary curve C.
- Let $\vec{F}$ be a vector field in $\mathbb{R}^3$, $S \subset \text{dom}\,\vec{F}$. w/ components that have continuous partial deriv.

Then, $\boxed{\int_C \vec{F}\cdot d\vec{r} = \iint_C \text{curl}\,\vec{F}\cdot d\vec{S}}$

$* \iint_S \vec{F}\cdot dS = \iint_S \vec{F}\cdot\hat{n}\, dS$

Special case: S is a planar region in xy-plane.

$\hat{n} = \hat{k}$ and $\iint_S \text{curl } \vec{F} \cdot d\vec{S} = \iint_S \text{curl } \vec{F} \cdot \hat{k} \, dS = \iint_S \text{curl } \vec{F} \cdot \hat{k} \, dA$

Stoke's theorm becomes:

$\int_C \vec{F} \cdot d\vec{r} = \iint_S \text{curl } \vec{F} \cdot \hat{k} \, dA = \iint_S Q_x - P_y \, dA$ ← green's thm.

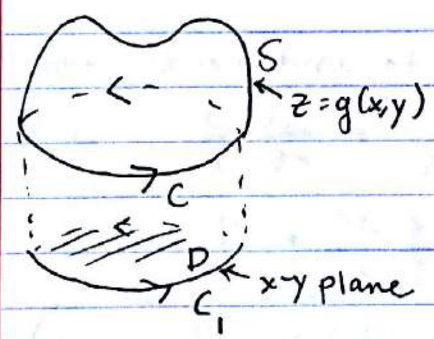
MATH: $\text{curl } \vec{F} = (R_y - Q_z)\hat{i} + (P_z - R_x)\hat{j} + (Q_x - P_y)\hat{k}$.

$\text{curl } \vec{F} \cdot \langle 0,0,1 \rangle = Q_x - P_y \, \hat{k}$.

Proof of Stoke's Theorem:

- Special case: Assume S is the graph $z = g(x,y)$ where g has a continuous 2nd partial deriv., $(x,y) \in D$ (simple plane region) whose boundary curve $C_1$ (in xy-plane) traces C, the boundary curve of S.

- The positive orientation of C is clockwise, like $C_1$.




$\iint_S \text{curl } \vec{F} \cdot d\vec{S}$ can be comp. with:

$\iint_D \left(-P \frac{\partial g}{\partial x} - Q \frac{\partial g}{\partial y} + R\right) dA = \iint_S \vec{F} \cdot d\vec{S}$

b/c S is graph of a func. (sec 16.7)

$\text{curl } \vec{F} = \left(\frac{\partial R}{\partial y} - \frac{\partial Q}{\partial z}\right)\hat{i} + \left(\frac{\partial P}{\partial z} - \frac{\partial R}{\partial x}\right)\hat{j} + \left(\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y}\right)\hat{k}$

So, $\iint_S \text{curl } \vec{F} \cdot d\vec{S} = \boxed{\iint_D \left[ -\left(\frac{\partial R}{\partial y} - \frac{\partial Q}{\partial z}\right)\frac{\partial z}{\partial x} - \left(\frac{\partial P}{\partial z} - \frac{\partial R}{\partial x}\right)\frac{\partial z}{\partial y} + \left(\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y}\right)\right] dA}$ ✱

$\int_C \vec{F}\cdot d\vec{r}$. $\quad C_1:\ x=x(t)\quad y=y(t)\quad a\le t\le b$

$C:\ x=x(t)\quad y=y(t)\quad z=g(x(t),y(t)),\ a\le t\le b.$

$$\text{So } \int_C \vec{F}\cdot d\vec{r} = \int_C P\,dx + Q\,dy + R\,dz$$

$$= \int_a^b \left(P\frac{dx}{dt} + Q\frac{dy}{dt} + R\frac{dz}{dt}\right)dt$$

$$= \int_a^b \left[P\frac{dx}{dt} + Q\frac{dy}{dt} + R\left(\frac{\partial z}{\partial x}\frac{dx}{dt} + \frac{\partial z}{\partial y}\frac{dy}{dt}\right)\right]dt$$

$$= \int_a^b \left[P + R\frac{\partial z}{\partial x}\right]\frac{dx}{dt} + \left[Q + R\frac{\partial z}{\partial y}\right]\frac{dy}{dt}\,]\,dt$$

$$= \int_{C_1} \left(P + R\frac{\partial z}{\partial x}\right)dx + \left(Q + R\frac{\partial z}{\partial y}\right)dy \text{ , by Green's,}$$

$$= \iint_D \left[\frac{\partial}{\partial x}\left(Q + R\frac{\partial z}{\partial y}\right) - \frac{\partial}{\partial y}\left(P + R\frac{\partial z}{\partial x}\right)\right]dA.$$

$$\frac{\partial}{\partial x}\left(Q + R\frac{\partial z}{\partial y}\right) = \frac{\partial}{\partial x}Q + \frac{\partial}{\partial x}R\frac{\partial z}{\partial y} = \frac{\partial}{\partial x}Q + \left(\frac{\partial}{\partial x}R\right)\frac{\partial z}{\partial y} + R\cdot\frac{\partial}{\partial x}\left(\frac{\partial z}{\partial y}\right)$$

by product rule.

Since $Q$ is a func of $x, y, z$ and $z$ is a func of $x, y$:

$$\frac{\partial}{\partial x}Q = \frac{\partial Q}{\partial x}\frac{\partial x}{\partial x} + \frac{\partial Q}{\partial y}\frac{\partial y}{\partial x} + \frac{\partial Q}{\partial z}\frac{\partial z}{\partial x} \quad \left(\frac{\partial x}{\partial x}=1,\ \frac{\partial y}{\partial x}=0\right)$$

$$= \frac{\partial Q}{\partial x} + \frac{\partial Q}{\partial z}\frac{\partial z}{\partial x}. \quad \text{So, } \frac{\partial}{\partial x}\left(Q + R\frac{\partial z}{\partial y}\right) =$$

$$= \underbrace{\frac{\partial Q}{\partial x} + \frac{\partial Q}{\partial z}\frac{\partial z}{\partial x}}_{\frac{\partial}{\partial x}Q} + \underbrace{\frac{\partial R}{\partial x}\frac{\partial z}{\partial y} + \frac{\partial R}{\partial z}\frac{\partial z}{\partial x}\frac{\partial z}{\partial y}}_{\left(\frac{\partial}{\partial x}R\right)\frac{\partial z}{\partial y}} + \underbrace{R\frac{\partial^2 z}{\partial x\partial y}}_{R\cdot\frac{\partial}{\partial x}\left(\frac{\partial z}{\partial y}\right).}$$

and $$\frac{\partial}{\partial y}\left(P + R\frac{\partial z}{\partial x}\right) = \frac{\partial P}{\partial y} + \frac{\partial P}{\partial z}\frac{\partial z}{\partial y} + \frac{\partial R}{\partial y}\frac{\partial z}{\partial x} + \frac{\partial R}{\partial z}\frac{\partial z}{\partial y}\frac{\partial z}{\partial x} + R\frac{\partial^2 z}{\partial y\partial x}.$$

So becomes $$\iint_D \left[\left(\frac{\partial Q}{\partial x} + \frac{\partial Q}{\partial z}\frac{\partial z}{\partial x} + \frac{\partial R}{\partial x}\frac{\partial z}{\partial y}\right) - \left(\frac{\partial P}{\partial y} + \frac{\partial P}{\partial z}\frac{\partial z}{\partial y} + \frac{\partial R}{\partial y}\frac{\partial z}{\partial x}\right)\right]dA$$

$$\checkmark \quad = \iint_D \left[-\left(\frac{\partial R}{\partial y} - \frac{\partial Q}{\partial z}\right)\frac{\partial z}{\partial x} - \left(\frac{\partial P}{\partial z} - \frac{\partial R}{\partial x}\right)\frac{\partial z}{\partial y} - \left(\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y}\right)\right]dA$$

for more general surfaces, do the partitioning thing.

Curl & Circulation:

Recall $\int_C \vec{F}\cdot d\vec{r} = \int_C \vec{F}\cdot\vec{T}\,ds$. ← unit tangent vector.

If $C$ is an oriented, closed curve, $\int_C \vec{F}\cdot d\vec{r}$ is called the circulation of $\vec{F}$ around $C$.

Let $P_0 = (x_0, y_0, z_0)$. Let $S_a$ be a small disk w/ center $P_0$ and radius $a$. Let $C_a$ be its boundary circle. Then,

$$\int_{C_a} \vec{F}\cdot d\vec{r} = \iint_{S_a} \text{curl}\,\vec{F}\cdot d\vec{S} \quad \text{by Stoke's theorem.}$$

$$\approx \iint_{S_a} \text{curl}\,\vec{F}(P_0)\cdot\hat{n}(P_0)\,dS$$

$$= \text{curl}\,\vec{F}(P_0)\cdot\hat{n}(P_0)\,\pi a^2. \quad \text{(gets better as } a\to 0.)$$

$$\text{So } \text{curl}\,\vec{F}\cdot\hat{n}(P_0) = \lim_{a\to 0}\frac{1}{\pi a^2}\int_{C_a}\vec{F}\cdot d\vec{r}.$$

- a measure of rotating effect about an axis $\vec{n}$.

Proof of 16.5.4:

If curl $\vec{F} = \vec{0}$ and dom $\vec{F}$ is open & simply connected, $\vec{F}$ is conservative.

Let $C$ be a closed path. By Stoke's theorem,

$$\int_C \vec{F}\cdot d\vec{r} = \iint_S \text{curl}\,\vec{F}\cdot d\vec{S} = \iint_S \vec{0}\cdot d\vec{S} = 0. \quad \text{Whee!}$$