11/23/15

16.7 Surface Integrals

Let $f(x,y,z)$ be a func. with dom $f \subset S$, where S is a parametric surface w/ eqn $\vec{r}(u,v) = x(u,v)\hat{i} + y(u,v)\hat{j} + z(u,v)\hat{k}$, $(u,v) \in \mathbf{D}$.

To define the surface integral of f over S,

- subdivide D into rectangles R_{ij} of sides $\Delta u, \Delta v. \Rightarrow S_{ij}$ of S.
- evaluate f at a sample point $P_{ij} \in S_{ij}$.
- Riemann Sum:
  $\sum_{i=1}\sum_{j=1} f(P_{ij}^*)\Delta S_{ij}$, where ΔS_{ij} is the area of S_{ij}.
- Take limit of $m, n \to \infty$.
- Recall $\Delta S_{ij} \approx |\vec{r}_u \times \vec{r}_v| \Delta u \Delta v$. ($\vec{r}_u$ and $\vec{r}_v$ are tangent vectors)

$$\therefore \boxed{\iint_S f(x,y,z)\, dS = \iint_D f(\vec{r}(u,v))\, |\vec{r}_u \times \vec{r}_v|\, dA.}$$

Note: if $f(x,y,z)$, $\iint_S 1\, dS = \iint_D |\vec{r}_u \times \vec{r}_v|\, dA = A(S)$!

Special Case: S is the graph of a function $g(x,y)$

parametrized: $z = g(x,y)$, $y = y$, $x = x$, so $\vec{r}_x = 1\hat{i} + 0\hat{j} + \frac{\partial g}{\partial x}\hat{k}$

$\vec{r}_y = 0\hat{i} + 1\hat{j} + \frac{\partial g}{\partial y}\hat{k}$.

So, $\vec{r}_x \times \vec{r}_y = -\frac{\partial g}{\partial x}\hat{i} - \frac{\partial g}{\partial y}\hat{j} + 1\hat{k}$

and $|\vec{r}_x \times \vec{r}_y| = \sqrt{1 + \left(\frac{\partial g}{\partial x}\right)^2 + \left(\frac{\partial g}{\partial y}\right)^2}$.

$$\therefore \iint_S f(x,y,z)\, dS = \iint_D f(x,y,g(x,y))\sqrt{1 + \left(\frac{\partial g}{\partial x}\right)^2 + \left(\frac{\partial g}{\partial y}\right)^2}\, dA.$$

If S is piecewise-smooth (finite union of smooth surfaces that only intersect on their boundaries, then

$$\iint_S f(x,y,z)\, dS = \sum_{i=1}^{n} \iint_{S_i} f(x,y,z)\, dS.$$

Mass and center of mass:

Suppose a thin sheet in the shape of surface S has density $\rho(x,y,z)$.

The total mass of the sheet is $\iint_S \rho(x,y,z)\, dS$.

center of mass: $(\bar{x}, \bar{y}, \bar{z})$ where $\bar{x} = \frac{1}{m}\iint_S x\rho(x,y,z)\, dS$

$$\bar{y} = \frac{1}{m}\iint_S y\rho(x,y,z)\, dS$$

$$\bar{z} = \frac{1}{m}\iint_S z\rho(x,y,z)\, dS.$$

Oriented Surfaces:

Let S be a surface that has a tangent plane at every pt. At each pt, there are 2 normal unit vectors: $\vec{n}_1$ and $\vec{n}_2 = -\vec{n}_1$. If it is possible to choose a unit normal vector $\vec{n}$ at each pt so that $\vec{n}$ varies continuously over S, then S is called an <u>orientable surface</u>. The given choice provides an orientation to S.

(non-orientable surfaces exist, like mobius strips).

- If S is a <u>closed</u> surface (boundary of some solid region), the positive orientation is taken to be the one w/ normal vectors pointing outward. The negative orientation has normal vectors pointing inward.
- If S is the graph of a function $g(x,y)$, the positive orientation is given by: $\vec{n} = \dfrac{-\frac{\partial g}{\partial x}\hat{i} + -\frac{\partial g}{\partial y}\hat{j} + 1k}{\sqrt{\left(\frac{\partial g}{\partial x}\right)^2 + \left(\frac{\partial g}{\partial y}\right)^2 + 1}}$ $\vec{n}$ points upward.
- If S is given by the vector function $\vec{r}(u,v)$, the orientation can be given by $\vec{n} = \dfrac{\vec{r}_u \times \vec{r}_v}{|\vec{r}_u \times \vec{r}_v|}$ (opposite orientation is $\vec{r}_v \times \vec{r}_u$).

Surface Integrals of Vector Fields:
let $\vec{F}$ be a continuous vector field on an oriented surface S w/ unit normal vector $\vec{n}$. The <u>surface integral of $\vec{F}$ over S</u> is:

$$\boxed{\iint_S \vec{F}\cdot d\vec{S} = \iint_S \vec{F}\cdot\vec{n}\, dS}$$

* also called flux of $\vec{F}$ accross S.

If S has vector function $\vec{r}(u,v)$, then $\iint_S \vec{F}\cdot d\vec{S}$

$$= \iint_S \vec{F}\cdot\vec{n}\,dS = \iint_S \vec{F}\cdot\frac{\vec{r}_u\times\vec{r}_v}{|\vec{r}_u\times\vec{r}_v|}\,dS = \iint_D \vec{F}\cdot\frac{\vec{r}_u\times\vec{r}_v}{|\vec{r}_u\times\vec{r}_v|}\,|\vec{r}_u\times\vec{r}_v|\,dA$$

Thus, $$\boxed{\iint_S \vec{F}\cdot\vec{n}\,dS = \iint_D \vec{F}\cdot(\vec{r}_u\times\vec{r}_v)\,dA}$$

$\uparrow$ $\vec{F}(\vec{r}(u,v))$

Special Case: S is the graph of a function $g(x,y)$
$u=x$, $v=y$. Recall $\vec{r}_x\times\vec{r}_y = -\frac{\partial g}{\partial x}\hat{\imath} - \frac{\partial g}{\partial y}\hat{\jmath} + 1\hat{k}$.

If $\vec{F} = P\hat{\imath} + Q\hat{\jmath} + R\hat{k}$, then $\vec{F}\cdot(\vec{r}_x\times\vec{r}_y) = -P\frac{\partial g}{\partial x}\hat{\imath} - Q\frac{\partial g}{\partial y}\hat{\jmath} + R\hat{k}$.

So $$\boxed{\iint_S \vec{F}\cdot\vec{n}\,dS = \iint_D \left(-P\frac{\partial g}{\partial x} - Q\frac{\partial g}{\partial y} + R\right)dA}$$

Remark: Formulas similar to the ones derived can be obtained for surfaces $y = h(x,z)$ or $x = k(y,z)$.