11/20/15

16.6 Parametric Surfaces and their Areas

Let $\vec{r}(u,v) = x(u,v)\,\hat{\imath} + y(u,v)\,\hat{\jmath} + z(u,v)\,\hat{k}$ be a vector function defined on a region D of the uv plane.

The set

$S = \{(x,y,z) \in \mathbb{R}^3 \mid x = x(u,v), y = y(u,v), z = z(u,v)$ for some $(u,v) \in D\}$

is a <u>parametric surface</u>. Its parametric eqns:

$x = x(u,v) \quad y = y(u,v) \quad z = z(u,v)$ ← 2 parameters.

As (u,v) varies in D, the tip of $\vec{r}(u,v)$ traces S.

* $\vec{r}(u,v)$ is <u>not</u> a vector field.

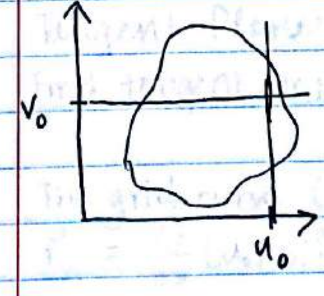
Grid Curves:

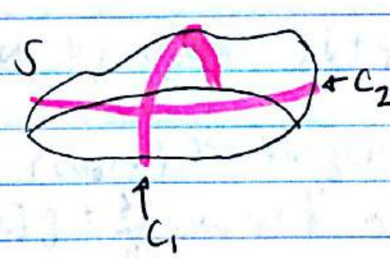
S, our parametric surface, has grid curves.

(2 families - 1 has u constant and the other, v).

- If $u = u_0$, $\vec{r}(u_0, v)$ is a vector that traces a curve C_1 lying on S.
- If $v = v_0$, $\vec{r}(u, v_0)$ traces a curve C_2 lying on S.

C_1 and C_2 intersect at $\vec{r}(u_0, v_0)$.





Grid curves are helpful for sketching parametric surfaces and for obtaining tangent planes.

Special Case: $z = f(x,y)$:

-x, y are equivalent of u, v.

- Domain is domain of f.
- parametric eqns: $x=x \quad y=y \quad z=f(x,y)$

$\vec{r}(x,y) = x\hat{i} + y\hat{j} + f(x,y)\hat{k}$.

The grid curves are the traces of S in the vertical planes $x=x_0$ and $y=y_0$!

Special Case: Surface of Revolution:

Let S be the surface obtained by rotating the cure $y=f(x)$, $a\le x \le b$ about the x-axis.

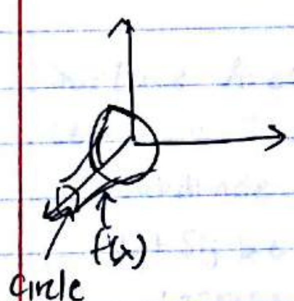
If θ is angle of rot, x and θ can be parameters.

So, parametric eqns: $x=x \quad y=f(x)\cos\theta \quad z=f(x)\sin\theta$

$a\le x\le b, \quad 0\le\theta\le 2\pi$

The grid cures $x=x_0$ are circles

$\theta=\theta_0$ are rotated images of cure $f(x)$



Tangent Planes:

Find tangent plane to S given by $\vec{r}(u,v) = xi + yj + zk$ at P_0 = tip of $\vec{r}(u_0,v_0)$

The grid cure C_1 given by $\vec{r}(u_0,v)$ has tangent vector

$\vec{r}_v = \frac{\partial x}{\partial x}(u_0,v_0)\hat{i} + \frac{\partial y}{\partial v}(u_0,v_0)\hat{j} + \frac{\partial z}{\partial v}(u_0,v_0)\hat{k}$ at P_0.

The grid curve C_2 given by $\vec{r}(u,v_0)$ has tangent vector

$\vec{r}_u = \frac{\partial x}{\partial u}(u_0,v_0)\hat{i} + \frac{\partial y}{\partial u}(u_0,v_0)\hat{j} + \frac{\partial z}{\partial u}(u_0,v_0)\hat{k}$ at P_0.

The tangent plane to S at P_0 has $\vec{r}_v$ and $\vec{r}_u$, so its normal vector is $\vec{r}_v \times \vec{r}_u$.

(assume $\neq \vec{0}$, or else S is smooth at P_0)

In the special case $z=f(x,y)$,

$x=x \quad y=y \quad z=f(x,y)$.

$\vec{r}_u = \vec{r}_x = \langle 1, 0, f_x(x_0,y_0)\rangle$

$\vec{r}_v = \vec{r}_y = \langle 0, 1, f_y(x_0,y_0)\rangle$

$$\vec{r}_u \times \vec{r}_v = \begin{vmatrix} \hat{\imath} & \hat{\jmath} & \hat{k} \\ 1 & 0 & f_x(x_0,y_0) \\ 0 & 1 & f_y(x_0,y_0 \end{vmatrix}$$

$\vec{n} = -f_x(x_0,y_0)\,\hat{\imath} - f_y(x_0,y_0)\,\hat{\jmath} + 1\,\hat{k}$.

and $\vec{n} = f_x(x_0,y_0)\,\hat{\imath} + f_y(x_0,y_0)\,\hat{\jmath} - 1\,\hat{k}$.

Equation of plane:

$$f_x(x_0,y_0)(x-x_0) + f_y(x_0,y_0)(y-y_0) - (z-z_0) = 0$$

Surface Area:

Let S be $\vec{r}(u,v) = x(u,v)\,\hat{\imath} + y(u,v)\,\hat{\jmath} + z(u,v)\,\hat{k}$, $(u,v)\in D$.

- Subdivide D into small rectangles R_{ij} with sides $\Delta u, \Delta v$
- Let S_{ij} be the image of R_{ij} (can be approx. with a parallelogram)
  - approx w/ $\Delta u \vec{r}_u$ and $\Delta v \vec{r}_v$. ($\vec{r}_u$ & $\vec{r}_v$ are tang. vectors)
- The area of the parallelogram is:

$|\Delta u \vec{r}_u \times \Delta v \vec{r}_v| = |\vec{r}_u \times \vec{r}_v| \Delta u \Delta v$.

$A(S) = \iint_D |\vec{r}_u \times \vec{r}_v|\, dA$ where $\vec{r}_u = \frac{\partial x}{\partial u}\hat{\imath} + \frac{\partial y}{\partial u}\hat{\jmath} + \frac{\partial z}{\partial u}\hat{k}$

$\vec{r}_v = \frac{\partial x}{\partial v}\hat{\imath} + \frac{\partial y}{\partial v}\hat{\jmath} + \frac{\partial z}{\partial v}\hat{k}$.

note: In the special case where $z=f(x,y)$, we saw $\vec{r}_u \times \vec{r}_v = \vec{r}_x \times \vec{r}_y$

$= -f_x\hat{\imath} + f_y\hat{\jmath} + 1\hat{k}$.

So, $A(S) = \iint_D \sqrt{1+\left(\frac{\partial z}{\partial x}\right)^2 + \left(\frac{\partial z}{\partial y}\right)^2}\, dA$. (sec. 15.6!)

note 2: In the special case where S is the surface of revolution obtained by rotating the curve $y = f(x)$ about the x-axis,

$x = x \quad y = f(x)\cos\theta \quad z = f(x)\sin\theta \quad a \le x \le b$
$0 \le \theta \le 2\pi.$

$\vec{r}_x = 1\hat{i} + f'(x)\cos\theta\hat{j} + f'(x)\sin\theta\hat{k}$ and
$\vec{r}_\theta = 0\hat{i} - f(x)\sin\theta\hat{j} + f(x)\cos\theta\hat{k}.$

$\vec{r}_x \times \vec{r}_\theta = f(x)f'(x)\hat{i} - f(x)\cos\theta\hat{j} - f(x)\sin\theta\hat{k}.$

$|\vec{r}_x \times \vec{r}_\theta| = f(x)\sqrt{1+[f'(x)]^2}.$ $A(S) = \iint_D f(x)\sqrt{1+f'(x)^2}\,dA$

$= \int_0^{2\pi}\int_a^b f(x)\sqrt{1+f'(x)^2}\,dx\,d\theta$

$$\boxed{A(S) = 2\pi\int_a^b f(x)\sqrt{1+f'(x)^2}\,dx}$$