16.5 Curl and Divergence

Curl: Let $\vec{F} = P\hat{i} + Q\hat{j} + R\hat{k}$ be a vector field on $\mathbb{R}^3$. The <u>curl of $\vec{F}$</u> is the vector field on $\mathbb{R}^3$ given by:

$$\boxed{\text{curl}\,\vec{F} = \left(\frac{\partial R}{\partial y} - \frac{\partial Q}{\partial z}\right)\hat{i} + \left(\frac{\partial P}{\partial z} - \frac{\partial R}{\partial z}\right)\hat{j} + \left(\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y}\right)\hat{k}.}$$

Notation: del operator

$\vec{\nabla} = \hat{i}\frac{\partial}{\partial x} + \hat{j}\frac{\partial}{\partial y} + \hat{k}\frac{\partial}{\partial z}$. ($\vec{\nabla}$ on scalar func. of 3 var. yields the gradient).

If you think of $\vec{\nabla}$ as a vector, $\boxed{\text{curl}\,\vec{F} = \vec{\nabla} \times \vec{F}}$

The curl of a gradient vector field is 0!

Proof:

$$\text{curl}(\vec{\nabla} f) = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ \partial/\partial x & \partial/\partial y & \partial/\partial z \\ \partial f/\partial x & \partial f/\partial y & \partial f/\partial z \end{vmatrix}$$

$$= \left(\frac{\partial^2 f}{\partial y \partial z} - \frac{\partial^2 f}{\partial z \partial y}\right)\hat{i} - \left(\frac{\partial^2 f}{\partial x \partial z} - \frac{\partial^2 f}{\partial z \partial x}\right)\hat{j} + \left(\frac{\partial^2 f}{\partial x \partial y} - \frac{\partial^2 f}{\partial y \partial x}\right)$$

$= 0$ by Clairaut's Theorem.

Theorem: If $\vec{F}$ is a vector field on $\mathbb{R}^3$ defined on a simply connected domain and having and having continuous partial derivatives, then if $\text{curl}\,\vec{F} = 0$, it is a <u>conservative field</u>. (It is also "irotational")

Divergence:

$\text{div}\,\vec{F} = \frac{\partial P}{\partial x} + \frac{\partial Q}{\partial y} + \frac{\partial R}{\partial z}$. $\text{div}\,\vec{F} = \vec{\nabla} \cdot \vec{F}$.

Theorem: If $\vec{F} = \langle P, Q, R \rangle$, and P, Q, R have continuous second partial derivatives, then $\text{div curl}\,\vec{F} = 0$.

Proof:

div curl F $= \vec{\nabla} \cdot (\vec{\nabla} \times \vec{F}) =$

$$\frac{\partial}{\partial x}\left(\frac{\partial R}{\partial y} - \frac{\partial Q}{\partial z}\right) + \frac{\partial}{\partial y}\left(\frac{\partial P}{\partial z} - \frac{\partial R}{\partial x}\right) + \frac{\partial}{\partial z}\left(\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y}\right)$$

$$= \boxed{\frac{\partial^2 R}{\partial x \partial y}} - \boxed{\frac{\partial^2 Q}{\partial x \partial z}} + \boxed{\frac{\partial^2 P}{\partial y \partial z}} - \boxed{\frac{\partial^2 R}{\partial y \partial x}} + \boxed{\frac{\partial^2 Q}{\partial z \partial x}} - \boxed{\frac{\partial^2 P}{\partial z \partial y}}.$$

$= 0$ by Clairaut's theorem.

Divergence describes the "fluid flow" → it measures the tendency of a fluid to diverge from a pt.
div $\vec{F} = 0$ means the fluid is "incompressible".

Laplace Operator, Laplace Equation:

$$\text{div}(\vec{\nabla} f) = \frac{\partial^2 f}{\partial x^2} + \frac{\partial^2 f}{\partial y^2} + \frac{\partial^2 f}{\partial z^2} = \vec{\nabla}^2(f).$$

$\vec{\nabla}^2 = \vec{\nabla} \cdot \vec{\nabla}$ ← Laplace operator

$\vec{\nabla}^2 = 0$ ← Laplace's Equation

* $\vec{\nabla}^2$ can be applied to a vector field $\vec{F}$.
$\vec{\nabla}^2(\vec{F}) = \vec{\nabla}^2 P \hat{i} + \vec{\nabla}^2 Q \hat{j} + \vec{\nabla}^2 R \hat{k}$

Vector Forms of Green's Theorem:

Let $\vec{F} = P\hat{i} + Q\hat{j}$. Let D be the region w/ a boundary curve $\partial D = C$ satisfying conditions of Green's Theorem.

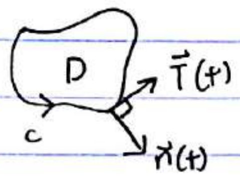
$\vec{F} = P\hat{i} + Q\hat{j} + 0\hat{k}$ on $\mathbb{R}^3$. curl $\vec{F} = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ \partial/\partial x & \partial/\partial y & \partial/\partial z \\ P & Q & 0 \end{vmatrix}$

$$= \left(\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y}\right)\hat{k}.$$

So... $\boxed{\oint \vec{F} \cdot d\vec{r} = \iint_D (\text{curl } \vec{F}) \cdot \hat{k} \, dA}$

Recall $\oint_C \vec{F}\cdot d\vec{r} = \oint \vec{F}\cdot\vec{T}\,ds$, $\vec{T} = \dfrac{\vec{r}\,'}{|\vec{r}\,'|}$

Let's use the <u>normal</u> component!



$C = \vec{r}(t) = x(t)\,\hat{\imath} + y(t)\,\hat{\jmath}$

$$\vec{T}(t) = \frac{\vec{r}\,'(t)}{|\vec{r}\,'(t)|} = \frac{x'(t)}{|r'(t)|}\hat{\imath} + \frac{y'(t)}{|r'(t)|}\hat{\jmath}$$

then the normal vector to C is:

unit vector $\rightarrow$ $$\vec{n}(t) = \frac{y'(t)}{|\vec{r}\,'(t)|}\hat{\imath} - \frac{x'(t)}{|r'(t)|}\hat{\jmath}. \quad (\vec{T}(t)\cdot\vec{n}(t) = 0)$$

Recall $\vec{F} = P(x,y)\,\hat{\imath} + Q(x,y)\,\hat{\jmath}$.

16.2 $\rightarrow$ $$\oint \vec{F}\cdot\vec{n}\,ds = \int_a^b (\vec{F}\cdot\vec{n})(t)\,|\vec{r}\,'(t)|\,dt$$

$$= \int_a^b \left[P(x(t),y(t))\frac{y'(t)}{|r'(t)|} - Q(x(t),y(t))\frac{x'(t)}{|r'(t)|}\right]|r'(t)|\,dt$$

$$= \int_a^b P\,dy - Q\,dx = \oint_C P\,dy - Q\,dx.$$

By Green's theorem, $= \iint_D \left(\frac{\partial P}{\partial x} + \frac{\partial Q}{\partial y}\right) dA.$

$$\boxed{\text{So, } \oint_C \vec{F}\cdot\vec{n}\,ds = \iint_D \operatorname{div}\vec{F}(x,y)\,dA.}$$