11/16/15

16.4 Green's Theorem

Orientation of Simple Closed curves:

- The <u>positive orientation</u> of a simple closed curve is obtained by traversing it once counterclockwise.
- If a region has one or more holes, its boundary will consist of two or more simple curves.

Region is to the left of direction curves are traced.

Green's Theorem:

C is a positively oriented smooth, simple closed curve. D is the region enclosed by C. If P & Q have continuous partial derivatives on an open region, that contains D, then

$$\boxed{\iint_D \left(\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y}\right) dA = \oint_C P\,dx + Q\,dy.}$$

$\int_C P\,dx + Q\,dy = \int_C \vec{F}\cdot d\vec{r}$, so Green's Theorem says:

$$\iint_D \left(\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y}\right) dA = \int_C \vec{F}\cdot d\vec{r}.$$

If $\vec{F}$ is conservative, $\frac{\partial Q}{\partial x} = \frac{\partial P}{\partial y}$, so $\iint_D \left(\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y}\right) dA = 0$.

and $\int_C \vec{F}\cdot d\vec{r} = 0$ (c is closed.)

Proof:

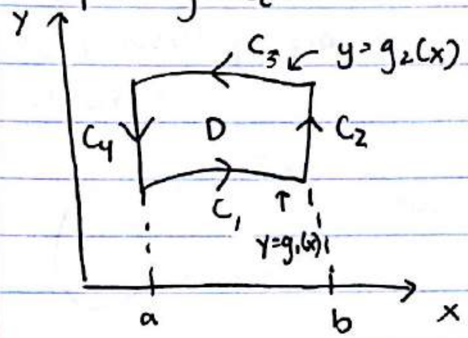
- A region is simple if it is type I and type II.

Case 1: D is simple. It can be desc. as $D = \{(x,y) \mid a \le x \le b, g_1(x) \le y \le g_2(x)$

So, $\iint_D \frac{\partial P}{\partial y}\,dA = \int_a^b \int_{g_1(x)}^{g_2(x)} \frac{\partial P}{\partial y}(x,y)\,dy\,dx$

$$= \int_a^b \left[P(x, g_2(x)) - P(x, g_1(x))\right] dx.$$

Computing $\int_C P\,dx$: $C = C_1 \cup C_2 \cup C_3 \cup C_4$



So $\int_C P\,dx = \int_{C_1} P\,dx + \int_{C_2} P\,dx + \int_{C_3} P\,dx + \int_{C_4} P\,dx$

On C_2 and C_4, x is constant, so
$\int_{C_2} P\,dx = \int_{C_4} P\,dx = 0$.

C_1 : $x = x$, $y = g_1(x)$, $a \le x \le b$. $\therefore$
$\int_{C_1} P\,dx = \int_a^b P(x, g_1(x))\,dx$

C_3 : $x = x$ $y = g_2(x)$, $a \le x \le b$ $\therefore$
$\int_{C_3} P\,dx = \int_a^b P(x, g_2(x))\,dx$

Therefore: $-\int_{-C_3} P\,dx = -\int_a^b P(x, g_2(x))\,dx$.

So $\int_C P\,dx = \int_a^b P(x, g_1(x))\,dx - \int_a^b P(x, g_2(x))\,dx$.

Thus $\int_C P\,dx = -\iint_D \frac{dP}{dy}\,dA$

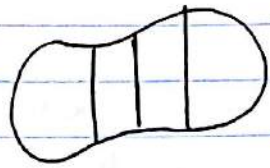
Do the same for D as a type II region to get:

$$\int_C Q\,dy = \iint \frac{\partial Q}{\partial x}\,dA.$$

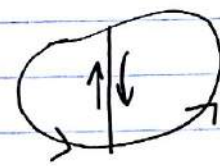
Add together and ...

$$\int_C P\,dx + Q\,dy = \iint_D \frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y}\,dA \checkmark$$

Case 2: D can be decomposed to a finite union of non-overlapping simple regions by drawing suitable lines or arcs.



You can apply Case 1 to each simple region.

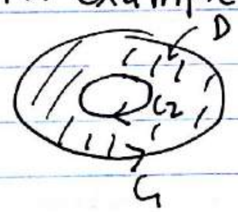


intersections cancel out.

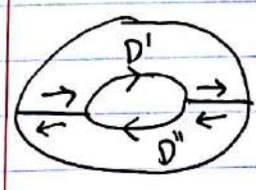
Case 3: D is any region enclosed by a simple closed curve. Then approx. D by regions & use a limit.

Green's Theorem for regions w/ Holes:
This applies, as long as you define the boundary correctly

For example:



D can be decomposed into two regions, each enclosed by a simple curve.



Since the intersections cancel out, Green's theorem works.

Application: Areas

$A(D) = \iint_D 1\, dA$. So, if $\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y} = 1$, then $A(D) = \iint_D \left(\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y}\right) dA = \int_C P\,dx + Q\,dy$.

$$\boxed{A(D) = \int_C x\,dy = -\int_C y\,dx = \frac{1}{2}\int_C x\,dy - y\,dx.}$$