11/08/15

16.2 Line Integrals

Flashback:

- Single Integrals over an integral $[a,b]$: $\int_a^b F(x)\,dx$ If $f>0$, this is the area between the curve and the x-axis (from $x=a$ to $x=b$).
- If C is a parametric curve with equations $x=x(t)$, $y=y(t)$, $a\leq t\leq b$, then the length of C is $L=\int_a^b \sqrt{\left(\frac{dx}{dt}\right)^2+\left(\frac{dy}{dt}\right)^2}\,dt = \int ds$ where $ds=\sqrt{\left(\frac{dx}{dt}\right)^2+\left(\frac{dy}{dt}\right)^2}\,dt$.
- The same argument in 3 dimensions shows that if C is given by: $x=x(t)$, $y=y(t)$, $z=z(t)$, $a\leq t\leq b$, then its length is $L=\int ds = \int_a^b \sqrt{\left(\frac{dx}{dt}\right)^2+\left(\frac{dy}{dt}\right)^2+\left(\frac{dz}{dt}\right)^2}\,dt$

Line Integrals:

Instead of integrating over an interval, we integrate a function over a curve in the plane or in space.

$\Rightarrow \int_C f\,ds$ instead of $\int_C 1\,ds \leftarrow$ arc/curve length

Line Integrals over a Curve in the Plane:

Let C be a curve given by parametric equations

*Assume C is smooth

$x=x(t)$, $y=y(t)$, $a\leq t\leq b$ a.k.a. $\vec{r}(t)=\langle x(t), y(t)\rangle = x(t)\hat{\imath}+y(t)\hat{\jmath}$

- Partition the parameter interval $[a,b]$ into n subintervals of equal width Δt: $a=t_0<t_1<\ldots<t_n=b$ $(\Delta t=t_i-t_{i-1})$
- Let $x_i=x(t_i)$, $y_i=y(t_i)$. The points $P_i=(x_i,y_i)$ subdivide C into lengths of ΔS_i.
- Choose a point $P_i^*=(x_i^*, y_i^*)$ where $x_i^*=x(t_i^*)$, $y=y(t_i^*)$
- Form the Riemann sum $\sum_{i=1}^{n} f(x_i^*, y_i^*)\,\Delta S_i$, take limit $n\to\infty$ to get line integral
- Notation: $\int_C f(x,y)\,ds$

If f is continuous, the limit always exists.

$$\boxed{\int_C f(x,y)\,ds = \int_a^b f(x(t),y(t))\underbrace{\sqrt{\left(\frac{dx}{dt}\right)^2+\left(\frac{dy}{dt}\right)^2}}_{\text{from 10.2}}\,dt}$$

This can be rewritten as:

$$\int_C f(x,y)\,ds = \boxed{\int_a^b f(\vec{r}(t))\,|\vec{r}\,'(t)|\,dt}$$

*make sure C is only traversed once!

Application: mass and c.o.M :

If a thin wire in the shape of a curve C has linear density $\rho(x,y)$, then the total mass of the wire is $m = \int_C \rho(x,y)\,ds$, and the center of mass is at $(\bar{x},\bar{y})$,

$\bar{x} = \frac{1}{m}\int_C x\rho(x,y)\,ds$ and $\bar{y} = \frac{1}{m}\int_C y\rho(x,y)\,ds$.

Line Integrals w.r.t. x and y:

The integral $\int_C f(x,y)\,ds$ is sometimes called the line integral of f along C with respect to arc length.

In the definition $\lim_{n\to\infty}\sum_{i=1}^{n} f(x_i^*,y_i^*)\Delta s_i$, if we replace Δs_i with $\Delta x_i = x_i - x_{i-1}$ and $\Delta y_i = y_i - y_{i-1}$, then it is called the line integrals of f along C with respect to x & y.

$\int_C f(x,y)\,dx = \int_a^b f(x(t),y(t))\,x'(t)\,dt$ and

$\int_C f(x,y)\,dy = \int_a^b f(x(t),y(t))\,y'(t)\,dt$.

Notation: Often line integrals with respect to x & y occur together. We write $\int_C P(x,y)\,dx + \int_C Q(x,y)\,dy = \int_C P(x,y)\,dx + Q(x,y)\,dy$

Orientation: A parametrization of C determines an orientation of C, with the pos. direction corresponding to increasing t. If $-C$ denotes the curve consisting of the same points but with the opposite orientation, we have:

$$\int_{-C} f(x,y)\,dx = -\int_C f(x,y)\,dx \quad \text{and} \quad \int_{-C} f(x,y)\,dy = -\int_C f(x,y)\,dy$$

because Δx_i and Δy_i change signs when the orientation is reversed. However, the line integral w/respect to arc length does not change. (Δs_i is pos.).

Line Integrals over a curve in Space:
If C is a smooth space curve with vector eqn
$\vec{r}(t) = \langle x(t), y(t), z(t)\rangle = x(t)\hat{i} + y(t)\hat{j} + z(t)\hat{k}$,
the line integral of f along C, with respect to arc length is quite similar:

$$\int_C f(x,y,z)\,ds = \int_a^b f(x(t),y(t),z(t))\sqrt{\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2 + \left(\frac{dz}{dt}\right)^2}\,dt.$$

$$= \int_a^b f(\vec{r}(t))\,|\vec{r}\,'(t)|\,dt$$

w/respect to x: $\int_C f(x,y,z)\,dx = \int_a^b f(\vec{r}(t))\,x'(t)\,dt$

y: $\int_C f(x,y,z)\,dy = \int_a^b f(\vec{r}(t))\,y'(t)\,dt$

z: $\int_C f(x,y,z)\,dz = \int_a^b f(\vec{r}(t))\,z'(t)\,dt$

$\int_C P(x,y,z)\,dx + Q(x,y,z)\,dy + R(x,y,z)\,dz$
means
$\int_C P(x,y,z)\,dx + \int_C Q(x,y,z)\,dy + \int_C R(x,y,z)\,dz$

Remark: You must parametrize curves when evaluating line integrals.
Recall that the vector eqn of the segment from $\vec{r}_0$ to $\vec{r}_1$ is:

$$\vec{r}(t) = (1-t)\vec{r}_0 + t\vec{r}_1, \quad 0 \le t \le 1$$

Line Integrals of Vector Fields:

- C is a smooth curve with vector eqn $\vec{r}(t)$, $a \le t \le b$
- $\vec{T}$ is the unit tangent vector of C. $\vec{T}(t) = \vec{r}\,'(t) \,/ |\vec{r}\,'(t)|$.
- $\vec{F}$ is a continuous vector field defined on C.

The line integral of $\vec{F}$ along C is:

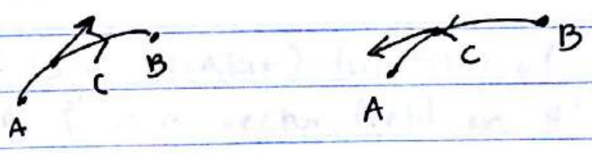
$$\int_C \vec{F} \cdot \vec{T}\, ds = \int_a^b \vec{F}(\vec{r}(t)) \cdot \vec{T}(t)\, |\vec{r}\,'(t)|\, dt$$

$\vec{T}(t) = \dfrac{\vec{r}\,'(t)}{|\vec{r}\,'(t)|}$ so, $\underbrace{\int_C \vec{F} \cdot \vec{T}\, ds}_{\text{often written as } \int_C F \cdot d\vec{r}} = \int_a^b \vec{F}(\vec{r}(t)) \cdot \vec{r}\,'(t)\, dt$

So, $$\boxed{\int_C F \cdot d\vec{r} = \int_C \vec{F}(\vec{r}(t)) \cdot \vec{r}\,'(t)\, dt = \int_C \vec{F} \cdot \vec{T}\, ds}$$

Work: If $\vec{F}$ is a force field, the work done by $\vec{F}$ in moving a particle along the curve C is $W = \int_C \vec{F} \cdot \vec{T}\, ds$.

Remark: $\int_C \vec{F} \cdot \vec{T} = \int_C \vec{F} \cdot d\vec{r}$ does change sign if the orientation is reversed, because $\vec{T}$ changes sign.



$$\int_{-C} \vec{F} \cdot \vec{T}\, ds = -\int_C \vec{F} \cdot \vec{T}\, ds$$