11/06/15

16.1 Vector Fields

Definitions:

- A <u>vector field on $\mathbb{R}^2$</u> is a function $\vec{F}: D \to V_2$ where $D \subset \mathbb{R}^2$. (D is a plane region, V_2 is a space of 2D vectors) If we write $\vec{F}(x,y) = P(x,y)\hat{\imath} + Q(x,y)\hat{\jmath}$, then P and Q are <u>component functions</u> of $\vec{F}$.
  Shorthand notation: $\vec{F} = P\hat{\imath} + Q\hat{\jmath} = \langle P, Q \rangle$

- A vector field on $\mathbb{R}^3$ is a function $\vec{F}: E \to V_3$ where $E \subset \mathbb{R}^3$. If we write $\vec{F}(x,y,z) = P(x,y,z)\hat{\imath} + Q(x,y,z)\hat{\jmath} + R(x,y,z)\hat{k}$, then P, Q, and R are the component functions of $\vec{F}$.
  Shorthand notation: $\vec{F} = P\hat{\imath} + Q\hat{\jmath} + R\hat{k} = \langle P, Q, R \rangle$

Continuity:

The vector field $\vec{F}: D \to V_2$ is continuous at $(a,b) \in D$ if $\lim_{(x,y)\to(a,b)} Q(x,y) = Q(a,b)$. So component functions are continuous @ (a,b).

Important Example: Gradient Fields:

If $f(x,y)$ is a function of 2 variables, then $\vec{\nabla} f$ is a vector field. Also, $\vec{\nabla} f = \langle f_x, f_y, f_z \rangle$ is a vector field (f is a func. of 3 variables)

Recall that at each point (x,y) or (x,y,z), the gradient vector $\vec{\nabla} f(x,y)$ or $\vec{\nabla} f(x,y,z)$ points in the direction of maximum increase ($\perp$ to level curves/surfaces).

Conservative Vector Fields:

A vector field $\vec{F}$ is <u>conservative</u> if it is the gradient of some scalar function f. That is, there exists a function f such that $\vec{F} = \vec{\nabla} f$. f is called a <u>potential function</u> for $\vec{F}$.

(yay, physics!)

Example: Newton's law of Gravitation.
If two objects with mass m and M are a distance r apart, then the magnitude of the gravitational force between them is $\frac{GMm}{r^2}$, where G is the gravitational constant.

Let the object with mass M be at the origin, and let $\vec{x} = \langle x, y, z \rangle$ be the position vector of the location (x, y, z) of the object with mass m.
Then, $r = |\vec{x}|$ and the gravitational force exerted on the object of mass m points toward the origin.
The unit vector in this direction is $\frac{-1}{|\vec{x}|}\vec{x}$.

The gravitational force is:
$$\vec{F}(\vec{x}) = \frac{GmM}{r^2} \cdot \left(\frac{-1}{|\vec{x}|}\vec{x}\right) = \frac{-GMm}{|\vec{x}|^3}\vec{x}.$$
This is called the gravitational field.

Component Functions:
$$\vec{F}(x,y,z) = \frac{-GMmx}{(x^2+y^2+z^2)^{3/2}}\hat{i} + \frac{-GMmy}{(x^2+y^2+z^2)^{3/2}}\hat{j} + \frac{-GMmz}{(x^2+y^2+z^2)^{3/2}}\hat{k}$$

Note $\vec{F}$ is a conservative field. Let $f(x,y,z) = \frac{mMG}{(x^2+y^2+z^2)^{1/2}}$

Then, it is clear that $\vec{\nabla} f = \langle f_x, f_y, f_z \rangle = \vec{F}$.

Example: Coulomb's Law
Place an electric charge Q at the origin. The electric force exerted by this charge on a charge q at (x,y,z) is:
$$\vec{F}(\vec{x}) = \frac{\varepsilon q Q}{|\vec{x}|^3}\vec{x},$$ where ε is a constant.

(similar to gravitational field.)

- If the charges have the same sign, the force is repulsive. $(qQ > 0)$

- If the charges have opposite signs the force is attractive $(qQ < 0)$, much like the grav. field.

- electric field of Q is the electric force per unit charge:

$$\vec{E}(\vec{x}) = \frac{1}{q}\vec{F}(\vec{x}) = \frac{\varepsilon Q}{|\vec{x}|^3}\vec{x}.$$

Remark: Preview:

In §16.3, we will see that line integrals of consecutive vector fields depend only on the initial and final points. Work done by a conservative force field in moving an object around a closed path is zero. We will see that the only vector fields that are "path independent" are the conservative vector fields.