11/02/15

15.10 Change of Variables in Multiple Integrals

Flashback:

$$\int_a^b f(x)\,dx = \int_c^d f(g(u))\,g'(u)\,du \qquad x = g(u) \quad c = g(a)$$
$$dx = g'(u)\,du \quad d = g(b)$$

Also written as $\int_a^b f(x)\,dx = \int_c^d f(x(u))\frac{dx}{du}\,du$

Transformations of $\mathbb{R}^2$:

Consider a change of variables given by a transformation T from the uv-plane to the xy-plane:

$T(u,v) = (x,y)$ where $x = g(u,v)$ & $y = h(u,v)$

* Assume g & h have continuous partial derivatives

T maps a region S in the uv-plane to a region R in the xy-plane. ($\partial S \to \partial R$)

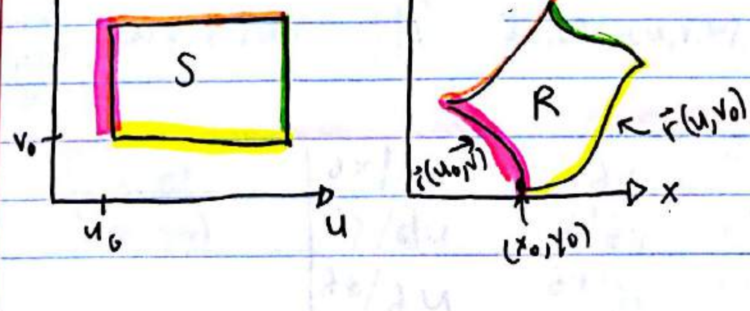
The Jacobian of T:

Let S be a small rectangle in the uv-plane with lower left corner (u_0, v_0) and sides Δu, Δv.

Let $R = T(S)$, the image of S, which is a region in the xy-plane. One of its boundary pts is $T(u_0,v_0) = (x_0, y_0)$

$\Rightarrow x_0 = g(u_0,v_0)$, $y_0 = h(u_0,v_0)$

The vector $\vec{r}(u,v) = g(u,v)\hat{\imath} + h(u,v)\hat{\jmath}$ is the position vector of R.



Tangent vector at (x_0, y_0) (lower side of S) is:

$\vec{r}_u = g_u(u_0,v_0)\hat{\imath} + h_u(u_0,v_0)\hat{\jmath}$

$= \boxed{\frac{\partial x}{\partial u}\hat{\imath} + \frac{\partial y}{\partial u}\hat{\jmath}}$

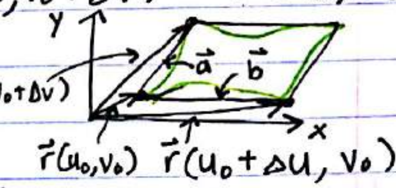
left side of S:

$\vec{r}_u = \boxed{\frac{dx}{dv}\hat{\imath} + \frac{dy}{dv}\hat{\jmath}}$

$\vec{r}_u$ = partial derivative w/ respect to u

Note that R can be approximated by a parallelogram with adjacent sides $\vec{a} = \vec{r}(u_0+\Delta u, v_0) - \vec{r}(u_0, v_0)$ and $\vec{b} = \vec{r}(u_0, v_0+\Delta v) - \vec{r}(u_0, v_0)$

But, $\vec{r}_u = \lim_{\Delta u \to 0} \dfrac{\vec{r}(u_0+\Delta u, v_0) - \vec{r}(u_0, v_0)}{\Delta u}$.



so $\vec{a} \approx \Delta u \vec{r}_u$. Similarily, $\vec{b} \approx \Delta v \vec{r}_v$

We can approximate R by the parallelogram with adjacent vectors $\Delta u \vec{r}_u$ and $\Delta v \vec{r}_v$. We do this by

$$|\Delta u \vec{r}_u \times \Delta v \vec{r}_v| = \Delta u \Delta v |\vec{r}_u \times \vec{r}_v|$$

So we have:

$$\vec{r}_u \times \vec{r}_v = \begin{vmatrix} \hat{i} & \hat{j} & k \\ \frac{\partial x}{\partial u} & \frac{\partial y}{\partial u} & 0 \\ \frac{\partial x}{\partial v} & \frac{\partial y}{\partial v} & 0 \end{vmatrix} = \left(\frac{\partial x}{\partial u}\frac{\partial y}{\partial v} - \frac{\partial x}{\partial v}\frac{\partial y}{\partial u} \right) \hat{k}$$

Note that $\begin{vmatrix} \frac{\partial x}{\partial u} & \frac{\partial y}{\partial u} \\ \frac{\partial x}{\partial v} & \frac{\partial y}{\partial v} \end{vmatrix} = \begin{vmatrix} \frac{\partial x}{\partial u} & \frac{\partial x}{\partial v} \\ \frac{\partial y}{\partial u} & \frac{\partial y}{\partial v} \end{vmatrix}$

This determinant is called the Jacobian of T.

It is denoted by $\dfrac{\partial(x,y)}{\partial(u,v)}$. So $A(R) \approx \left| \dfrac{\partial(x,y)}{\partial(u,v)} \right| \Delta u \Delta v$.

Change of Variables for Double Integrals:

$$\boxed{\iint_R f(x,y)\, dA = \iint_S f(x(u,v), y(u,v)) \left| \frac{\partial(x,y)}{\partial(u,v)} \right| du\, dv}$$

Change of Variables for Triple Integrals:

$$\boxed{\iiint_R f(x,y,z)\, dV = \iiint_S f(x(u,v,w), y(u,v,w), z(u,v,w)) \left| \frac{\partial(x,y,z)}{\partial(u,v,w)} \right| du\, dv\, dw}$$

$$\frac{\partial(x,y,z)}{\partial(u,v,w)} = \begin{vmatrix} \partial x/\partial u & \partial x/\partial v & \partial x/\partial w \\ \partial y/\partial u & \partial y/\partial v & \partial y/\partial w \\ \partial z/\partial u & \partial z/\partial v & \partial z/\partial w \end{vmatrix}$$

Special Case: Integration in Polar Coordinates:

$x = r\cos\theta$
$y = r\sin\theta$

Jacobian: $\begin{vmatrix} \partial x/\partial r & \partial x/\partial\theta \\ \partial y/\partial r & \partial y/\partial\theta \end{vmatrix}$

$= \begin{vmatrix} \cos\theta & -r\sin\theta \\ \sin\theta & r\cos\theta \end{vmatrix}$

$= r\cos^2\theta + r\sin^2\theta = \boxed{r}$

$$\iint_R f(x,y)\, dA = \iint_S f(r\cos\theta, r\sin\theta)\, r\, dr\, d\theta$$!

Special Case: Integration in Cylindrical Coordinates:

$x = r\cos\theta$
$y = r\sin\theta$
$z = z$

Jacobian: $\begin{vmatrix} \partial x/\partial r & \partial x/\partial\theta & \partial x/\partial z \\ \partial y/\partial r & \partial y/\partial\theta & \partial y/\partial z \\ \partial z/\partial r & \partial z/\partial\theta & \partial z/\partial z \end{vmatrix}$

$= \begin{vmatrix} \cos\theta & -r\sin\theta & 0 \\ \sin\theta & r\cos\theta & 0 \\ 0 & 0 & 1 \end{vmatrix} = \boxed{r}$

$$\iiint_R f(x,y,z)\, dV = \iiint_S f(r\cos\theta, r\sin\theta, z)\, r\, dz\, dr\, d\theta$$!

Special Case: Integration in Spherical Coordinates

$x = \rho\sin\phi\cos\theta$
$y = \rho\sin\phi\sin\theta$
$z = \rho\cos\phi$

Jacobian: $\begin{vmatrix} \partial x/\partial\rho & \partial x/\partial\theta & \partial x/\partial\phi \\ \partial y/\partial\rho & \partial y/\partial\theta & \partial y/\partial\phi \\ \partial z/\partial\rho & \partial z/\partial\theta & \partial z/\partial\phi \end{vmatrix}$

$= \begin{vmatrix} \sin\phi\cos\theta & -\rho\sin\phi\sin\theta & \rho\cos\phi\cos\theta \\ \sin\phi\sin\theta & \rho\sin\phi\cos\theta & \rho\cos\phi\sin\theta \\ \cos\phi & 0 & -\rho\sin\phi \end{vmatrix}$

$= -\rho^2\sin\phi, \quad |-\rho^2\sin\phi| = \underline{\rho^2\sin\phi}.$

$$\iiint_R f(x,y,z)\, dV = \iiint_S f(\rho\sin\phi\cos\theta, \rho\sin\phi\sin\theta, \rho\cos\phi)\, \rho^2\sin\phi\, d\rho\, d\theta\, d\phi.$$

!

Special Case: Linear Transformations of the Plane

Shift & Switch:

$x = au + k$ ← stretches by fact. of a in horizont. direc.

$y = bv + l$ ← stretches by fact. of b in vertical direc.

so we should expect the areas to be $\times |ab|$

Jacobian: $\begin{vmatrix} a & 0 \\ 0 & b \end{vmatrix} = ab$, $\iint_R f(x,y)\,dA = \iint_S f(au+k, bv+l)\,|ab|\,du\,dv$

Special case: Linear Transformations of Space-Shifts & Stretches

$x = au + k$

$y = bv + l$ Jacobian: $\begin{vmatrix} a & 0 & 0 \\ 0 & b & 0 \\ 0 & 0 & c \end{vmatrix} = abc$

$z = cw + m$

$$\iiint_R f(x,y,z)\,dV = \iiint_S f(au+k, bv+l, cw+m)\,|abc|\,du\,dv\,dw.$$