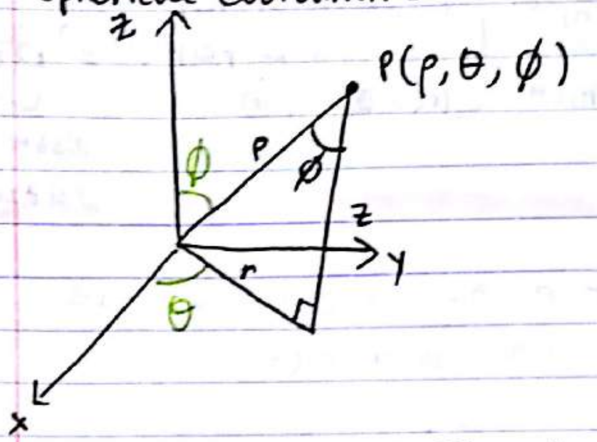
10/30/15

15.9 Triple Integrals in Spherical Coordinates

Spherical Coordinates:



ρ is the distance from origin to P
θ is the angle we saw in cylindrical coordinates
ϕ is the angle between the positive z-axis and the line from the origin to Point P.

Deriving Conversion Formulas:
r, z, and ρ form a right triangle. We can see that the angle between ρ and z is ϕ. Thus,

$$\boxed{\begin{aligned} r &= \rho \sin\phi \\ z &= \rho\cos\phi \end{aligned}}$$

Also, by the distance formula, $\boxed{\rho^2 = r^2 + z^2}$

we know $x = r\cos\theta$, $y = r\sin\theta$, so:

$$\boxed{\begin{aligned} x &= \rho\sin\phi\cos\theta \\ y &= \rho\sin\phi\sin\theta \end{aligned}} \longrightarrow \boxed{\cos\theta = \frac{x}{\rho\sin\phi}, \quad \sin\phi = \frac{y}{\rho\sin\theta}}$$

Evaluating Triple Integrals w/ Spherical Coordinates
The equivalent of a rectangular box in the spherical coordinate system is a "spherical wedge":

$$E = \{(\rho, \theta, \phi) \mid a \le \rho \le b, \alpha \le \theta \le \beta, c \le \phi \le d\}$$

NOTE: $\rho > 0$, and $d - c \le \pi$.

You get smaller wedges by subdividing $[a,b]$, $[\alpha, \beta]$, and $[c,d]$ into equal subintervals. Then we get subwedges E_{ijk} w/ approx volume

$$\Delta V_{ijk} \approx (\Delta\rho)(\rho_i \Delta\phi)(\rho_i \sin\phi_k \Delta\theta)$$

$$= \rho_i^2 \sin\phi_k \Delta\rho\, \Delta\theta\, \Delta\phi$$

Then you do the Riemann thing & get,

$$\iiint_E f(x,y,z)\,dV = \int_c^d \int_\alpha^\beta \int_a^b f(\rho\sin\phi\cos\theta, \rho\sin\phi\sin\theta, \rho\cos\phi)\,\rho^2\sin\phi\,d\rho\,d\theta\,d\phi$$

Example: what is the graph of $\rho = \csc\phi$? - no θ, so symmetric about z-axis.

$x^2+y^2 = r^2 = \rho^2\sin^2\phi$. Since $\rho = \csc\phi$, we get $x^2+y^2=1$.
(cylinder of $r=1$)

Example #40: Evaluate $\int_{-a}^{a}\int_{-\sqrt{a^2-y^2}}^{\sqrt{a^2-y^2}}\int_{-\sqrt{a^2-x^2-y^2}}^{\sqrt{a^2-x^2-y^2}} x^2z + y^2z + z^3 \,dz\,dx\,dy$

this region of integration is the sphere $x^2+y^2+z^2 \le a$.
So in spherical coord.,

$$\in \{(\rho,\theta,\phi) \mid 0 \le \rho \le a,\ 0 \le \theta \le 2\pi,\ 0 \le \phi \le \pi\}$$

And $x^2z + y^2z\, z^3 = \rho^2 z = \rho^3\cos\phi$

we get: $\int_0^\pi \int_0^{2\pi} \int_0^a \rho^5 \sin\phi\cos\phi \,d\rho\,d\theta\,d\phi$

$$= \int_0^\pi \sin\phi\cos\phi\,d\phi \int_0^{2\pi} d\theta \int_0^a \rho^5\,d\rho$$

$$= \left[\tfrac{1}{2}\sin^2\phi\right]_0^\pi \cdot 2\pi \cdot \tfrac{1}{6}a^6$$

$$= \boxed{0}$$